

1. i **False.** A linear system with more equations than unknowns **may or may not** have a solution. If the system is inconsistent it will have no solution. Therefore such a system is **not always** consistent.
- ii **False.** If the reduced echelon form of the coefficient matrix has a pivot in the last column, it is possible, for instance, that the last row is a zero row. Therefore, a unique solution is not guaranteed in every case.
- iii **False.** If $\mathbf{u}$ and $\mathbf{v}$ are solutions to the linear system, then any linear combination (including $\mathbf{u} + \mathbf{v}$) will **only** be a solution if $\mathbf{b} = \mathbf{0}$.
- iv **False.** In this case we have a linear system with more unknowns than equations. However, the system might be inconsistent.

2. From row operations we find the following forms

$$\left(\begin{array}{ccc|c} 0 & 5 & -5 & b_1 \\ 3 & 1 & 8 & b_2 \\ -3 & -4 & -5 & b_3 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 1/3 & 8/3 & b_2/3 \\ 0 & 1 & -1 & b_1/5 \\ 0 & 0 & 0 & 3b_1/5 + b_2 + b_3 \end{array} \right)$$

The system is consistent for $3b_1 + 5b_2 + 5b_3 = 0$ or $b_1 = -\frac{5}{3}(b_2 + b_3)$ or equivalent.

3. We first simplify the expression into $(A^T)^T - B^T + C(C^{-1}B) = A - B^T + IB$.
Because B is symmetric, we have the simplification as : $A - B + B = A$.

4. We perform row operations so that $(B|I) \rightarrow (I|B^{-1})$. We find the following inverse matrix:

Version A: $B^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 2 & -1 \\ 2 & 3 & -1 \end{pmatrix}$

Version B: $B^{-1} = \begin{pmatrix} -3 & -2 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & -1 \end{pmatrix}$

Version C: $B^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ -3 & -5 & 1 \\ 3 & 4 & -1 \end{pmatrix}$

Version D: $B^{-1} = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$

5. (a) The reduced echelon form of A is:

$$A \sim \dots \sim R = \begin{pmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\text{Col } A = \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

Any set of two, three or four of the four column vectors of A is acceptable.

(b) The reduced echelon form of A is :

$$A \sim \dots \sim R = \begin{pmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \text{ Hence the linearly independent vectors are the columns}$$

of the original matrix corresponding to the pivot elements. These are: $\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \right\}$ or, any set of the two of the four column vectors of A is acceptable because of linear independence.

6. Let's denote the given vectors by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ respectively. The vectors are linearly independent if and only if: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$ implies that $c_1 = c_2 = c_3 = 0$.

We write the vectors in a matrix form and perform row operations to bring the matrix to its reduced echelon form to search for linear independent vectors.

$$\begin{pmatrix} 1 & \alpha & 1 \\ 0 & 1 & 2\alpha \\ 0 & -\alpha & 3\alpha + 1 \end{pmatrix} \sim \dots \sim \begin{pmatrix} 1 & 0 & 1 - 2\alpha^2 \\ 0 & 1 & 2\alpha \\ 0 & 0 & 2\alpha^2 + 3\alpha + 1 \end{pmatrix},$$

The vectors are linearly independent for $2\alpha^2 + 3\alpha + 1 \neq 0$, that is $\alpha \neq -1$ or $\alpha \neq -\frac{1}{2}$.

7. If we take the initial matrix as:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

and perform the following elementary row operations:

- $R_1 \rightarrow xR_1$,
- $R_2 \rightarrow xR_2$,
- $R_2 \rightarrow R_2 - R_1$,
- $R_1 \rightarrow R_1 + R_3$,

then we arrive to the matrix:

$$B = \begin{pmatrix} ax + g & bx + h & cx + i \\ x(d - a) & x(e - b) & x(f - c) \\ g & h & i \end{pmatrix}$$

Only the first and the second of the elementary row operations defined above change the determinant (each as the product of x).

Therefore we can write: $\det A = x^2 \det B$, where $x \neq 0$.

8. To solve the problem, we need to show that $\lambda^3 - 2\lambda + 3$ is an eigenvalue of $A^3 - 2A + 3I$, where λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$ (where $\mathbf{v} \neq 0$).

Given that $A\mathbf{v} = \lambda\mathbf{v}$,

$$\begin{aligned} (A^3 - 2A + 3I)\mathbf{v} &= A^3\mathbf{v} - 2A\mathbf{v} + 3I\mathbf{v} \\ &= A^2(A\mathbf{v}) - 2(A\mathbf{v}) + 3\mathbf{v} \\ &= A^2(\lambda\mathbf{v}) - 2\lambda\mathbf{v} + 3\mathbf{v} \\ &= A(A(\lambda\mathbf{v})) - 2\lambda\mathbf{v} + 3\mathbf{v} \end{aligned}$$

$$\begin{aligned}
&= A(\lambda(\lambda\mathbf{v})) - 2\lambda\mathbf{v} + 3\mathbf{v} = A(\lambda^2\mathbf{v}) - 2\lambda\mathbf{v} + 3\mathbf{v} \\
&= \lambda^3\mathbf{v} - 2\lambda\mathbf{v} + 3\mathbf{v} \\
&= (\lambda^3 - 2\lambda + 3)\mathbf{v}
\end{aligned}$$

Hence $(A^3 - 2A + 3I)\mathbf{v} = (\lambda^3 - 2\lambda + 3)\mathbf{v}$. Therefore, $\lambda^3 - 2\lambda + 3$ is an eigenvalue of $A^3 - 2A + 3I$.

9. This question can be solved with a few approaches.

Method 1) Realizing that A has three distinct eigenvalues and the corresponding eigenvectors (let's denote by e_1, e_0, e_{-1}) form a basis for $\mathbb{R}^3$ and that we can express the vector $\begin{pmatrix} -3 \\ 2 \\ 16 \end{pmatrix}$ as a linear combination of e_1, e_0, e_{-1} . Finding the linear combination and using the property : $A^{10}\mathbf{v} = \lambda^{10}\mathbf{v} = \lambda^{10}(c_1e_1 + c_0e_0 + c_{-1}e_{-1})$

Method 2) Finding the P, D, P^{-1} matrices and using the property: $A = PDP^{-1}$ and then determining A^{10} by $A^{10} = PD^{10}P^{-1}$ and working out $A^{10}\mathbf{v} = PD^{10}P^{-1}\mathbf{v}$.

Method 3) Realizing that $A^3 = A$ and $A^10 = A^2$ and finding $A^{10}\mathbf{v} = A^2\mathbf{v}$.

Please note that there was a typo in the eigenvector: $\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$.

If you took this given eigenvector the solution is:

$$\begin{aligned}
\mathbf{v} &= \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \\ 7 \end{pmatrix}. \\
A^{10}\mathbf{v} &= A^{10} \left(\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \\ 7 \end{pmatrix} \right) \\
&= A^{10} \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} + A^{10} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + A^{10} \begin{pmatrix} -4 \\ 2 \\ 7 \end{pmatrix} \\
&= (-1)^{10} \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} - 0^{10} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + 1^{10} \begin{pmatrix} -4 \\ 2 \\ 7 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 12 \end{pmatrix}.
\end{aligned}$$

The correct eigenvector should have been $\begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix}$. In this case the solution for the correct vector is as follows:

$$P = \begin{pmatrix} 3 & -2 & -4 \\ -1 & 1 & 2 \\ -5 & 4 & 7 \end{pmatrix}, D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D^{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix}$$

$$PD^{10}P^{-1}\mathbf{v} = \begin{pmatrix} 7 & -2 & 4 \\ -3 & 2 & -2 \\ -12 & 4 & -7 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 16 \end{pmatrix} = \begin{pmatrix} 39 \\ -19 \\ -68 \end{pmatrix}.$$

10. **Version A:** We write the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ as a linear combination of the given vector,

$\begin{pmatrix} 5 \\ 6 \\ -13 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$. We solve the system and find that $c_1 = -2$, $c_2 = 3$. Since the transformation is linear we can use

$$T \begin{pmatrix} 5 \\ 6 \\ -13 \end{pmatrix} = -2T \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + 3T \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = -2 \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} \Rightarrow T \begin{pmatrix} 5 \\ 6 \\ -13 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \\ 13 \end{pmatrix}.$$

Version B: $c_1 = 2$, $c_2 = 3$

$$T \begin{pmatrix} 13 \\ 6 \\ 7 \end{pmatrix} = 2T \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + 3T \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 19 \\ 19 \end{pmatrix}.$$

Version C: $c_1 = -1$, $c_2 = 2$

$$T \begin{pmatrix} 5 \\ 6 \\ -13 \end{pmatrix} = -T \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + 2T \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix} = - \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 8 \end{pmatrix}.$$

Version D: $c_1 = 3$, $c_2 = 2$

$$T \begin{pmatrix} 9 \\ 6 \\ -1 \end{pmatrix} = 3T \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + 2T \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix} = 3 \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 16 \\ 16 \end{pmatrix}.$$

11. Given that $\mathbf{y}_1, \mathbf{y}_2$ are both in the image of T , show that $a\mathbf{y}_1 + b\mathbf{y}_2$ is also in $\text{im}(T)$ for all $a, b \in \mathbb{R}$.

Let $\mathbf{x}_3 = a\mathbf{x}_1 + b\mathbf{x}_2$ where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ such that $T(\mathbf{x}_1) = \mathbf{y}_1$ and $T(\mathbf{x}_2) = \mathbf{y}_2$

Because of linearity we can write:

$$T(\mathbf{x}_3) = T(a\mathbf{x}_1 + b\mathbf{x}_2) = T(a\mathbf{x}_1) + T(b\mathbf{x}_2) = aT(\mathbf{x}_1) + bT(\mathbf{x}_2) = a\mathbf{y}_1 + b\mathbf{y}_2$$

Thus, $a\mathbf{y}_1 + b\mathbf{y}_2$ is in $\text{im}(T)$ since there exists $\mathbf{x}_3 \in \mathbb{R}^n$ such that $T(\mathbf{x}_3) = a\mathbf{y}_1 + b\mathbf{y}_2$.