

Fluid Mechanics, Module 5

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1 Forces on a Body

There are 3 different forces on immersed bodies, but we'll focus on the first 2 forces:

1. Friction drag, due to contact with the surface (also known as skin friction).
2. Pressure drag, due to pressure differences (which are caused by flow separation)
3. Lift, orthogonal to the flow

A streamline can have 2 different states, turbulent and laminar.

The formula for Friction drag is:

$$F_D = C_D \cdot \frac{1}{2} \rho u^2 \cdot A \quad (1)$$

Where c_D is the drag coefficient, ρ is the density, u is the velocity and A is the area which is in touch with the water (or air).

The drag coefficient is really dependant on the Reynolds number and is not a fixed value. The Reynolds number (also written as Re), indicates how laminar or turbulent a flow is, higher is more turbulent. The Reynolds number can be calculated by:

$$Re = \frac{UD}{\nu} \quad (2)$$

Where U is the velocity, D is the length scale and ν is the *kinematic viscosity*. The C_d can be calculated by using the Reynolds number.

- For $5 \cdot 10^5 < Re < 10^7$, use equation:

$$C_D = \frac{0.0742}{Re^{1/5}}$$

- For $Re < 10^9$, use equation:

$$C_D = \frac{0.455}{(\log Re)^{2.58}}$$

2 Conservation of mass

The conservation of mass law says that the internal change of mass in a control volume + the net in/outflow through the boundaries of a control volume are zero.

2.1 Internal change of mass

Since we don't know the exact dimensions of this control volume, we can take a very small piece and assume that its mass is dM . since mass is volume multiplied by its density, we can say:

$$dM = \rho \cdot dVol \quad (3)$$

If we want the total mass within our CV (control volume), we can take the integral over dM :

$$M = \int dM = \int_{CV} \rho \cdot dVol \quad (4)$$

If we want to calculate our mass change per timestep Δt , the following formula can be used:

$$\frac{M_{t+\Delta t} - M_t}{\Delta t} = \frac{\partial M}{\partial t} = \frac{\partial (\int_{CV} \rho \cdot dVol)}{\partial t} \quad (5)$$

2.2 Net in/outflow of mass through boundaries

Now we are going to look at the net in and outflow of a surface element of the CV. Lets assume that through the surface element dA a fluid flows with velocity V under an angle θ . Now, we are able to derive a formula.

The outflow of mass in timestep Δt : $\rho \Delta x \cdot |d\vec{A}|$

With $\Delta x = |\vec{V}| \cos \theta \cdot \Delta t$.

This results in

$$\rho |\vec{V}| \cos \theta \cdot \Delta t \cdot |d\vec{A}| \quad (6)$$

If we want to do this for the whole system, we use the integral. By writing $|d\vec{A}| |\vec{V}| \cos \theta$ as $\rho \vec{V} \cdot d\vec{A}$, we can setup the integral from the boundaries of CV (= Control Surface, CS):

$$\int_{CS} \rho \vec{V} \cdot d\vec{A} \quad (7)$$

2.3 Conservation of mass

By adding formula 3 (internal change of mass) and formula 5 (mass transport through boundaries), we can make the **Integral mass balance**:

$$0 = \frac{\partial}{\partial t} \int_{CV} \rho dVol + \int_{CS} \rho \vec{V} \cdot d\vec{A} \quad (8)$$

For solving questions, a general procedure can be followed:

1. Draw control volume
2. Determine integral law and write it down
3. Write down assumptions (and how to apply) and define positive direction
4. Write down all the steps to come to your answer

3 Conservation of Momentum

The momentum balance P for a system is written as

$$\vec{P} = m\vec{V} \quad (9)$$

Knowing the amount of force and the length of time that force is applied to an object will tell you the resulting change in its momentum. They are related by the fact that force is the rate at which momentum changes with respect to time $F = dp/dt$.

Note that if $P = mV$ and m is constant, then $F = dp/dt = m \frac{dv}{dt} = ma$.

This gives us the final equation:

$$\sum \vec{F} = \frac{d\vec{P}}{dt} \quad (10)$$

3.1 Internal change of momentum

For a balance, we need to know the internal change of the system. This is build up from multiple equations. First we start with the mass. For this, we can use formula 3: $dM = \rho \cdot dVol$. filling this in into equation 10, this results into:

$$d\vec{P} = \vec{V}dM = \vec{V}\rho dVol \quad (11)$$

Since we want to know the total momentum in our CV, we need to take the integral of $d\vec{P}$:

$$\int d\vec{P} = \int_{CV} \rho \vec{V} dVol \quad (12)$$

Since we want to know the change of momentum per timestep (Δt), tho only thing left is to take the partial derivative of $\vec{P}$ over t :

$$\frac{\partial \vec{P}}{\partial t} = \frac{\partial}{\partial t} \int_{CV} \rho \vec{V} dVol \quad (13)$$

The formula we got is the **internal change of momentum**, which is bigger then zero when there is an increase.

3.2 Net in/outflow of momentum trough boundaries

The outflowing momentum per timestep is derived from the following formula's

$$\begin{aligned} Vol\rho\vec{V} &= \Delta x dA\vec{V} \\ \Delta x &= V \cos \theta \Delta t \end{aligned}$$

Combining these two equations, we get:

$$\begin{aligned} \rho|V| \cos \theta |dA| \Delta t \vec{V} \\ \vec{V} \rho \vec{V} \bullet d\vec{A} \\ \vec{a} \bullet \vec{b} &= |a||b| \cos \theta \end{aligned}$$

So for the entire surface CS of the control volume, the outflowing momentum per unit of time is:

$$\int_{CS} \vec{V} \rho \vec{V} \cdot d\vec{A} \quad (14)$$

3.3 Conservation of momentum

If we combine the formula's of the internal change of momentum and the net in/outflow, we get one big equation which follows the rule of conversation of momentum:

$$\sum \vec{F} = \vec{F}_s + \vec{F}_v = \frac{\partial}{\partial t} \int_{CV} \rho \vec{V} dVol + \int_{CS} \vec{V} \rho \vec{V} \cdot d\vec{A} \quad (15)$$

The conversation of momentum is valid in all three dimensions, so this gives us three equations in three directions. This results in:

$$\vec{V} = (U, V, W) \quad (16)$$

so when this is in the x -direction, we get a formula where the first $\vec{V}$ is replaced by a velocity in the U direction:

$$\sum \vec{F}_x = \frac{\partial}{\partial t} \int_{CV} U \rho dVol + \int_{CS} U \rho \vec{V} \cdot d\vec{A} \quad (17)$$

The same can be done in the y - and z -direction

4 Open channel flow

This part cannot be found in the book, found on canvas.

4.1 Open channel characteristics

There are 3 main questions asked regarding rivers:

1. Discharge: how much water can flow through a given profile
2. Water level: How high is the water level at maximum discharge? (important for building dikes)
3. Water depth: how low is the water level during minimal discharge (important for boats)

The hydraulic radius is important for the characteristics of an open channel. the hydraulic radius can be calculated by:

$$R = \frac{A}{P} \quad (18)$$

Where P is the wet perimeter in m and A is the flow area in m^2 . The wet perimeter is the length of the wet banks (so the soil underwater) when you have a cross-section. For a rectangular profile for example the equation changes to:

$$R = \frac{A}{P} = \frac{B \cdot h}{B + 2h} \quad (19)$$

If the width B is much bigger than the height h , R will be approximately equal to h .

It takes a while before increases in discharge can be noticed downstream, this is due the propagation speed of waves:

$$c = \pm \sqrt{gh} \quad (20)$$

This is a very important equation for determining *supercritical* and *subcritical* flow. Comparing the propagation speed with the velocity (u), the Froude number equation can be used:

$$Fr = \frac{u}{\sqrt{gh}} \quad (21)$$

... and the following can be stated:

- subcritical flow: $u < \sqrt{gh}$, $Fr < 1$. This is a typical Dutch situation, Weirs have an influence of what is happening upstream.
- supercritical flow: $u > \sqrt{gh}$, $Fr > 1$. This is mostly seen in steep mountain rivers, weirs are not useful.

4.2 Small-scale applications

The law of Bernoulli is mostly used when friction is not an big factor. The equation states:

$$\frac{p}{\rho} + gz + \frac{V^2}{2g} = const. \quad (22)$$

This can be rewritten to:

$$y + Z_b + \frac{V^2}{2g} = const. = H_0 \quad (23)$$

Where, y is the water height, Z_b is the height of the bottom of the river and V is the velocity,

4.3 Large-scale applications

The law of Chezy is an equation used for a turbulent, steady, uniform flow over a bed with a slope. With a flow over a bed with a slope, the $\sum F = 0$, and the two forces acting on the system are the gravity and the friction force, these forces should balance. The gravity force in the direction of the water bed can be calculated by:

$$F_z = m \cdot g \cdot \sin \beta = \rho \cdot h \cdot \delta x g \cdot \sin \beta$$

$$F_w = \tau_b \cdot \delta x$$

Since these forces are equal to each other, we can derive the τ_b :

$$\tau_b = \rho \cdot g \cdot h \cdot \sin \beta \approx \rho \cdot g \cdot h \cdot \tan \beta$$

$$= \rho \cdot g \cdot h \cdot S_b$$

$$S_b = \tan \beta$$

Now the Chezy equation begins playing a role:

$$V = C\sqrt{RS_B} \quad (24)$$

Where V is the velocity, C is the smoothness coefficient (bigger = smoother), R is the hydraulic radius and S_b is the bed slope. Since the hydraulic radius is not important with bed shear stress, it can be replaced by the water height (h). Rewriting gives us the follow formula for bed shear stress:

$$\tau_b = \rho \cdot g \cdot \frac{V^2}{C^2} \quad (25)$$

The C value can be derived on two ways:

1. Nikuradse roughness k , where k is the length/size of the surface (for example with vegetation, the length of the vegetation itself):

$$C = 18 \log \frac{12R}{k} \quad (26)$$

2. Manning roughness n , where n can be found in a table with all known surfaces (mostly used on the test):

$$C = \frac{R^{1/6}}{n} \quad (27)$$

Units: $C = [\frac{\sqrt{m}}{s}]$, $k = m$ and $n = [m^{-1/3}s]$

By using the manning coefficient, an useful relationship can be derived, the **Q-h relationship**:

$$Q = B \cdot h^{5/3} \frac{\sqrt{S_b}}{n} \quad (28)$$

Written with h as output variable:

$$h = \left[\frac{Q \cdot n}{B \cdot \sqrt{S_b}} \right]^{3/5} \quad (29)$$

The following equation is derived from equation 24 and 27, and is used a lot in exercises:

$$Q = A(R)^{2/3} \frac{\sqrt{S_b}}{n} \quad (30)$$

5 Backwater curves

A steady flow with gradually varying depth is an stationary non-uniform flow. backwater curves are functions of height h over the distance x , so $h(x)$. The equation is as follows:

$$\frac{\partial h}{\partial x} = S_b \frac{h^3 - h_n^3}{h^3 - h_c^3}$$

$$h_n = \left(\frac{q^2}{C^2 S_b} \right)^{1/3} \quad \text{Normal depth, (uniform flow)}$$

$$h_c = \left(\frac{q^2}{g} \right)^{1/3} \quad \text{Critical depth, (Fr = 1)}$$

In the test, mostly no calculations, but drawings have to be made. This is done by giving the slope i , the normal depth compared to the critical depth and the Froude number. If

- $i_b < i_c$, the bed slope is smaller then the critical slope, which is a **mild slope**.
- $i_b > i_c$, the bed slope is bigger then the critical slope, which is a **steep slope**.
- $i_b = i_c$, the bed slope is equal to the critical slope, which is a **critical slope**.

In case of:

- $Fr < 1$, the water height h is *bigger* then the critical depth h_c . This is *subcritical* flow. The boundary $x = x_0$ lies downstream.
- $Fr > 1$, the water height h is *smaller* then the critical depth h_c . This is *supercritical* flow. The boundary $x = x_0$ lies upstream.

On the test, an table with corresponding curves will be given. The curve can be found by using the possible properties mentioned above:

Let op: $h=h_0$ en $h_s=h_c$!
Tabel staat niet in de syllabus !

	zone 1: $h > h_c$ $h > h_0$	zone 2: $h_c < h < h_0$ $h_s < h < h_c$	zone 3: $h < h_c$ $h < h_0$
negative slope			
horizontal bed			
positive slope	mild slope $h_0 > h_c$		
	critical slope $h_0 = h_c$		
	steep slope $h_0 < h_c$		

Figure 1: Example of such table