

Exam Mathematical Physics of Water Systems

Course code: 195400900, 17 April 2014, 13:45-17:15, in Sportcentrum

Name:

Student ID:

This exam consists of 16 numbered items, worth a total of 10 points. It is *not* allowed to use the syllabus, textbooks or personal notes. The use of a pocket calculator is allowed. Please use this yellow answer sheet to answer the multiple choice items (and item 15); use a separate sheet for the other items. Good luck!

Answer sheet

Example: ☐ (a) ☒ (b) if alternative b is the correct answer.

(0.3p) 1. ☐ (a) uncoupled ☐ (b) reservoir I flows out in II ☐ (c) reservoir II flows out in I

(1.0p) 2. Use a separate sheet.

(0.3p) 3. ☐ (a) PDE ☐ (b) relaxation equation ☐ (c) equation for damped oscillator

(0.3p) 4. ☐ (a) parabolic ☐ (b) hyperbolic ☐ (c) elliptic ☐ (d) depends on γ and μ .

(0.3p) 5. ☐ (a) linear and homogeneous ☐ (b) linear and nonhomogeneous ☐ (c) nonlinear.

(0.3p) 6. ☐ (a) ☐ (b) ☐ (c) ☐ (d) ☐ (e)

(1.0p) 7. Use a separate sheet.

(0.5p) 8. Use a separate sheet.

(1.2p) 9. Use a separate sheet.

(0.3p) 10. ☐ (a) $Pe < 1$, $Mp < 1$ ☐ (b) $Pe < 1$, $Mp > 1$ ☐ (c) $Pe > 1$, $Mp < 1$ ☐ (d) $Pe > 1$, $Mp > 1$

(1.2p) 11. Use a separate sheet.

(0.3p) 12. ☐ (a) $T_2 < T_1$ ☐ (b) $T_2 = T_1$ ☐ (c) $T_2 > T_1$

(1.2p) 13. Use a separate sheet.

(0.3p) 14. Use a separate sheet.

(0.3p) 15. Provide the physical unit of β in this box:

(1.2p) 16. Use a separate sheet.

end of answer sheet <

Exam Mathematical Physics of Water Systems

Course code: 195400900, 17 April 2014, 13:45-17:15, in Sportcentrum

This exam consists of 16 numbered items, worth a total of 10 points. It is *not* allowed to use the syllabus, textbooks or personal notes. The use of a pocket calculator is allowed. Please use the yellow answer sheet to answer the multiple choice items (and item 15); use a separate sheet for the other items. Good luck!

Two reservoirs (1.6p)

Consider two linear reservoirs: reservoir I with storage volume $V(t)$ and reservoir II with storage volume $W(t)$. For each reservoir, we assume linear outflow relationships, with parameters a and b . As it turns out, the evolution of $V(t)$ and $W(t)$ satisfies the following differential problem:

$$\frac{dV}{dt} + aV = f, \quad \frac{dW}{dt} + bW = aV, \quad V(0) = V_0, \quad W(0) = W_0. \quad (1)$$

Here f is a time-dependent forcing term, and V_0 and W_0 are the initial storage volumes (at $t = 0$).

(0.3p) 1. From Eq.(1), what can you say about the coupling between these reservoirs?

- | | | |
|---------------|---------------------------------|---------------------------------|
| (a) uncoupled | (b) reservoir I flows out in II | (c) reservoir II flows out in I |
|---------------|---------------------------------|---------------------------------|

By eliminating V from Eq.(1), it can be shown that W satisfies the following differential problem:

$$\frac{d^2W}{dt^2} + p\frac{dW}{dt} + qW = r, \quad W(0) = W_0, \quad \frac{dW}{dt}(0) = s. \quad (2)$$

with parameters p, q (both positive), forcing term r and parameter s .

(1.0p) 2. Derive the differential problem in Eq.(2) and find expressions for p, q, r, s .

(0.3p) 3. How can you best describe the differential equation in Eq.(2)?

- | | | |
|---------|-------------------------|------------------------------------|
| (a) PDE | (b) relaxation equation | (c) equation for damped oscillator |
|---------|-------------------------|------------------------------------|

turn page for items 4-16 >

Advection-diffusion-growth problem

Model formulation (0.9p)

In an elongated pool of length L , the mass concentration $c(x, t)$ (in g m^{-3}) of a certain type of bacteria satisfies the following differential problem:

$$\frac{\partial c}{\partial t} + \gamma \frac{\partial c}{\partial x} = \delta \frac{\partial^2 c}{\partial x^2} + \mu, \quad (3)$$

Here, t is time (in s), x is the spatial coordinate (x in m, $0 < x < L$). Furthermore, we have introduced the constants γ (zero or positive), δ (strictly positive) and production term μ (zero or positive). The boundary conditions and initial conditions are given by

$$c(0, t) = c(L, t) = 0, \quad c(x, 0) = C [\sin(\pi x/L) + \epsilon \sin(2\pi x/L)]. \quad (4)$$

respectively. Here, C and ϵ are parameters.

(0.3p) 4. Of what type is the PDE in Eq.(3)?

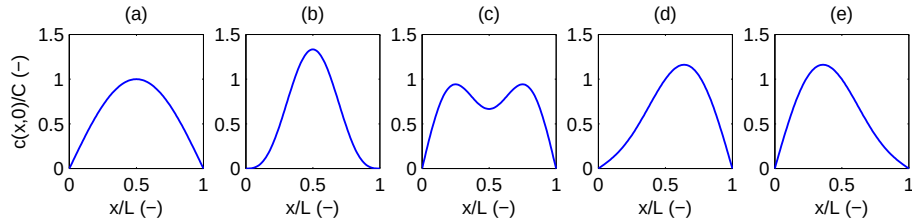
- ☐ (a) parabolic ☐ (b) hyperbolic ☐ (c) elliptic ☐ (d) depends on γ and μ .

(0.3p) 5. For $\mu > 0$, how can you further classify the PDE in Eq.(3)?

- ☐ (a) linear and homogeneous ☐ (b) linear and nonhomogeneous ☐ (c) nonlinear.

(0.3p) 6. For $\epsilon = \frac{1}{3}$, which of the five plots below correctly displays the initial profile?

- ☐ (a) ☐ (b) ☐ (c) ☐ (d) ☐ (e)



Equilibrium state without advection (1.5p)

Next, we study the existence and form of static equilibrium states. We do this for the case without advection, i.e. for $\gamma = 0$ (keeping $\delta > 0$ and $\mu > 0$).

(1.0p) 7. Derive an expression for the equilibrium profile $c_{\text{eq}}(x)$ without advection.

(0.5p) 8. From a physical perspective, what can you say about the diffusive fluxes across the boundaries in this equilibrium?

Scaling procedure if diffusion is dominant (1.5p)

A so-called scaling procedure helps to assess the relative importance of the various physical mechanisms involved. Assuming that *diffusion* is the dominant mechanism, we introduce the diffusive time scale $T_{\text{diff}} = L^2/\delta$ and define

$$c^* = \frac{c}{C}, \quad x^* = \frac{x}{L}, \quad t^* = \frac{t}{T_{\text{diff}}}. \quad (5)$$

items 9-16 on next pages >

It can be shown that the scaled form of Eq.(3) is given by

$$\frac{\partial c^*}{\partial t^*} + \boxed{\text{Pe}} \frac{\partial c^*}{\partial x^*} = \frac{\partial^2 c^*}{\partial x^{*2}} + \boxed{\text{Mp}}, \quad (6)$$

with *Péclet number* Pe and a new *Mpows number* Mp , to be specified below.

(1.2p) **9.** By applying the intended scaling procedure, derive Eq.(6), as well as expressions for Pe and Mp .

(0.3p) **10.** Based on the information given here, what can you say about the values of Pe and Mp ?

(a) $\text{Pe} < 1, \text{Mp} < 1$	(b) $\text{Pe} < 1, \text{Mp} > 1$	(c) $\text{Pe} > 1, \text{Mp} < 1$	(d) $\text{Pe} > 1, \text{Mp} > 1$
------------------------------------	------------------------------------	------------------------------------	------------------------------------

Analytical solution for diffusion only (1.5p)

Assuming that diffusion is the only mechanism, i.e. for $\gamma = 0$ and $\mu = 0$, a suitable trial solution to the problem in Eqs.(3)-(4) is given by

$$c(x, t) = C [f_1(t) \sin(\pi x/L) + f_2(t) \sin(2\pi x/L)], \quad (7)$$

with unknown functions $f_1(t)$ and $f_2(t)$. It can be shown that each of these functions must satisfy a homogenous relaxation problem.

(1.2p) **11.** Derive the homogeneous relaxation problems for $f_1(t)$ and $f_2(t)$.

The solutions to these relaxation problems are given by

$$f_1(t) = \exp(-t/T_1), \quad f_2(t) = \epsilon \exp(-t/T_2), \quad (8)$$

with so-called e-folding time scales T_1 and T_2 .

(0.3p) **12.** What can you say about T_2 compared to T_1 ?

(a) $T_2 < T_1$	(b) $T_2 = T_1$	(c) $T_2 > T_1$
-----------------	-----------------	-----------------

Numerical solution (1.5p)

To proceed numerically, let $c_{j,k}$ denote the numerical solution at $x = x_j = j\Delta x$ and $t = t_k = k\Delta t$ with numerical space step Δx and time step Δt . The index ranges are given by $j = 0, 1, 2, \dots, J$ (such that $x_J = L$) and $k = 0, 1, 2, \dots, K$. Next, Eq.(3) is discretised using a Forward Euler discretization of the time derivative, an Upwind discretisation of the first order space derivative and a familiar second order Central-Difference discretisation of the second-order space derivative. This leads to a so-called *explicit scheme* that can be written in the following form:

$$c_{j,k+1} = Pc_{j-1,k} + Qc_{j,k} + Rc_{j+1,k} + S, \quad (9)$$

with coefficients P , Q , R and S . It is possible to express P , Q and R in familiar dimensionless parameters known as the *Courant number* $\mathcal{C}$ and the *diffusion number* $\mathcal{D}$.

(1.2p) **13.** Express P , Q , R in dimensionless numbers, and also find an expression for S .

(0.3p) **14.** Sketch the numerical stencil if, instead of Forward Euler, we would apply Backward Euler to discretise the time derivative in Eq.(3).

turn page for items 15-16 >

Nonlinear growth model (1.5p)

Finally, we consider a different problem, namely that of the growth a certain species on the seabed. Let $f(t)$ denote the density in the physical unit of kg m^{-2} , which satisfies the following nonlinear differential problem:

$$\frac{df}{dt} = \beta f^{3/2}, \quad f(0) = f_0. \quad (10)$$

Here, β and f_0 are positive parameters ($\mu > 0$ and $f_0 > 0$), their precise values being unknown.

(0.3p) **15.** Derive the physical unit of β .

(1.2p) **16.** Using separation of variables for ODEs, find the solution $f(t)$.

end of exam <

Answers

Item 1 (0.3p)

(b) reservoir I flows out in II [0.3p]

Item 2 (1.0p)

The correct answer consists of the following steps:

- Take the time derivative of the W -equation [0.1p], then substitute the V -equation in this result [0.1p], and then eliminate V by using the W -equation again [0.1p]:

$$\frac{d^2W}{dt^2} + b\frac{dW}{dt} = a\frac{dV}{dt} = a[f(t) - aV] = af(t) - a\left[\frac{dW}{dt} + bW\right]. \quad (11)$$

- Rearrange to find $\frac{d^2W}{dt^2} + [a + b]\frac{dW}{dt} + abW = af(t)$ [0.1p].
- From this last result identify: $p = a + b$, $q = ab$ and $r(t) = af(t)$ [0.3p].
- The IC for $W(t)$ was already specified in Eq.(2): $W(0) = W_0$ [0.1p].
- A second IC follows from considering the ODE in Eq.(2) at $t = 0$ and applying the initial conditions for V and W . This gives $\frac{dW}{dt}(0) = aV_0 - bW_0$ showing that $s = aV_0 - bW_0$ [0.2p].

Item 3 (0.3p)

(c) equation for damped oscillator [0.3p]

Item 4 (0.3p)

(a) parabolic [0.3p]

Item 5 (0.3p)

(b) linear and nonhomogeneous [0.3p]

Item 6 (0.3p)

(e) [0.3p]

Item 7 (1.0p)

The correct answer consists of the following steps

- Equilibrium state c_{eq} is characterized by vanishing time derivative [0.1p].
- Must thus satisfy $0 = \delta \frac{d^2 c_{eq}}{dx^2} + \mu$ [0.1p].

- Integrating twice gives: $\delta c_{\text{eq}}(x) = -\frac{1}{2}\mu x^2 + ax + b$ with constants a and b [0.4p].
- Applying boundary conditions gives $b = 0$ [0.2p] and $a = \frac{1}{2}\mu L$ [0.2p], so final solution reads $c_{\text{eq}}(x) = \frac{\mu x(L-x)}{2\delta}$.

Item 8 (0.5p)

- In this equilibrium state, the fluxes across the boundaries of the domain must compensate the internal production so that the total amount of matter is conserved [0.2p].
- Hence, they must be directed outward, i.e. to the left at $x = 0$ and to the right at $x = L$ [0.2p].
- Moreover, by symmetry (production is uniform in the domain and there is no advection), these fluxes must be equal in magnitude [0.1p].

Item 9 (1.2p)

The correct answer consists of the following steps.

- Time derivative transforms to $\frac{\partial c}{\partial t} = \frac{C}{T_{\text{diff}}} \frac{\partial c^*}{\partial t^*}$ [0.1p]
- First order space derivative transforms to $\frac{\partial c}{\partial x} = \frac{C}{L} \frac{\partial c^*}{\partial x^*}$ [0.1p]
- Second order space derivative transforms to $\frac{\partial^2 c}{\partial x^2} = \frac{C}{L^2} \frac{\partial^2 c^*}{\partial x^{*2}}$ [0.1p]
- Substitution in PDE gives the following intermediate result [0.1p]:

$$\frac{C}{T_{\text{diff}}} \frac{\partial c^*}{\partial t^*} + \frac{\gamma C}{L} \frac{\partial c^*}{\partial x^*} = \frac{\delta C}{L^2} \frac{\partial^2 c^*}{\partial x^{*2}} + \mu. \quad (12)$$

- Rearranging this, with the aid of $T_{\text{diff}} = L^2/\delta$, gives the following dimensionless equation [0.2p]

$$\frac{\partial c^*}{\partial t^*} + \left[\frac{\gamma L}{\delta} \right] \frac{\partial c^*}{\partial x^*} = \frac{\partial^2 c^*}{\partial x^{*2}} + \left[\frac{\mu L^2}{\delta C} \right]. \quad (13)$$

- Here, we recognise *Péclet number* $\text{Pe} = \gamma L/\delta$ [0.3p] and the *Mpows number* $\text{Mp} = \mu L^2/(\delta C)$ [0.3p].

Item 10 (0.3p)

(a) $\text{Pe} < 1, \text{Mp} < 1$ [0.3p]

Item 11 (1.2p)

The correct answer consists of the following steps.

- Trial solution must be substituted in the PDE without advection and without production term **[0.1p]**.
- To this end, we need the time derivative **[0.2p]** and the second space derivative **[0.2p]**:

$$\frac{\partial c}{\partial t} = C[f_1'(t) \sin(\pi x/L) + f_2'(t) \sin(2\pi x/L)], \quad (14)$$

$$\frac{\partial^2 c}{\partial x^2} = -C[f_1(t)(\pi/L)^2 \sin(\pi x/L) + f_2(t)(2\pi/L)^2 \sin(2\pi x/L)]. \quad (15)$$

- Substitution in PDE now gives the following intermediate result **[0.1p]**:

$$C[\underbrace{f_1' + (\pi/L)^2 f_1}] \sin(\pi x/L) + C[\underbrace{f_2' + (2\pi/L)^2 f_2}] \sin(2\pi x/L) = 0. \quad (16)$$

- This equation must be satisfied at any x (and at any time) so each of the underbraced terms must individually be zero **[0.2p]**.
- Hence, we get two separate ODEs for $f_1(t)$ and $f_2(t)$, which indeed turn out to be homogeneous relaxation equations **[0.2p]**:

$$\frac{df_1}{dt} + (\pi/L)^2 f_1 = 0, \quad \frac{df_2}{dt} + (2\pi/L)^2 f_2 = 0. \quad (17)$$

- To complete the derivation of these two relaxation problems, the initial conditions finally follow from comparing the form of the trial solution and the initial profile: $f_1(0) = 1$ **[0.1p]** and $f_2(0) = \epsilon$ **[0.1p]**

Item 12 (0.3p)

$$\boxed{(a) \ T_2 < T_1} \quad \mathbf{[0.3p]}$$

Item 13 (1.2p)

The correct answer implies a discretisation of the PDE at $(x, t) = (x_j, t_k)$ **[0.1p]**.

- Forward Euler of the time derivative: $\frac{\partial c}{\partial t} \Big|_{(x_j, t_k)} \approx \frac{c_{j,k+1} - c_{j,k}}{\Delta t}$ **[0.1p]**.
- Upwind of the first order space derivative: $\frac{\partial c}{\partial x} \Big|_{(x_j, t_k)} \approx \frac{c_{j,k} - c_{j-1,k}}{\Delta x}$ **[0.1p]**.
- Central discr. of the 2nd space derivative: $\frac{\partial^2 c}{\partial x^2} \Big|_{(x_j, t_k)} \approx \frac{c_{j+1,k} - 2c_{j,k} + c_{j-1,k}}{[\Delta x]^2}$ **[0.1p]**.
- Substitution in the PDE then gives the following intermediate result **[0.1p]**:

$$\frac{c_{j,k+1} - c_{j,k}}{\Delta t} + \gamma \frac{c_{j,k} - c_{j-1,k}}{\Delta x} = \delta \frac{c_{j+1,k} - 2c_{j,k} + c_{j-1,k}}{[\Delta x]^2} + \mu \quad (18)$$

- After some rearrangements this gives the following result **[0.1p]**:

$$c_{j,k+1} = \left[\frac{\gamma\Delta t}{\Delta x} + \frac{\delta\Delta t}{[\Delta x]^2} \right] c_{j-1,k} + \left[1 - \frac{\gamma\Delta t}{\Delta x} - \frac{2\delta\Delta t}{[\Delta x]^2} \right] c_{j,k} + \left[\frac{\delta\Delta t}{[\Delta x]^2} \right] c_{j+1,k} + \mu\Delta t \quad (19)$$

- Comparison with Eq.(9) finally gives us, using the definitions of the *Courant number* $\mathcal{C} = \gamma\Delta t/\Delta x$ **[0.1p]** and the *diffusion number* $\mathcal{D} = \delta\Delta t/[\Delta x]^2$ **[0.1p]**, the following expressions for the coefficients **[0.4p]**:

$$P = \mathcal{C} + \mathcal{D}, \quad Q = 1 - \mathcal{C} - 2\mathcal{D}, \quad R = \mathcal{D}, \quad S = \mu\Delta t. \quad (20)$$

Item 14 (0.3p)

The plot should show the familiar numerical stencil as also used in the BTCS scheme for the diffusion **[0.2p]**. Make sure that also the meanings of space and time are indicated in the figure **[0.1p]**.

Item 15 (0.3p)

The physical unit of β is $\text{m s}^{-1} \text{kg}^{-1/2}$ **[0.3p]**.

Item 16 (1.2p)

The correct answer consists of the following steps.

- Separate variables and add integration **[0.2p]**: $\int \frac{df}{f^{3/2}} = \int \beta dt$.
- Perform integration on both left-hand **[0.2p]** and right-hand side **[0.2p]**, which should give one integration constant **[0.1p]**: $-2f(t)^{-1/2} = \beta t + C$.
- Apply IC to determine constant C **[0.2p]**: $-2f(t)^{-1/2} = \beta t - 2f_0^{-1/2}$.
- Rearrange to find the solution **[0.3p]**: $f(t) = \frac{1}{[f_0^{-1/2} - \frac{1}{2}\beta t]^2}$.