

**LINEAR ALGEBRA<sub>A</sub>** Date : April 26, 2024  
Time : 13.45 – 15.45 hrs

1. (2 pts) *Fill in your final answer to this exercise on the supplied answer sheet.*

For the system of equations with augmented matrix shown below, give the solution in parametric vector form. version A:

$$\left( \begin{array}{ccccc|c} 1 & 2 & -1 & 0 & -2 & 2 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\text{Answer A: } \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + k \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + l \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad k, l \in \mathbb{R} \text{ or equivalently } \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

version B:

$$\left( \begin{array}{ccccc|c} 1 & 0 & 2 & 1 & -2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\text{Answer B: } \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + k \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + l \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad k, l \in \mathbb{R} \text{ or equivalently } \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

version C:

$$\left( \begin{array}{ccccc|c} 1 & 3 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\text{Answer C: } \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} + k \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + l \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad k, l \in \mathbb{R} \text{ or equivalently } \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

version D:

$$\left( \begin{array}{ccccc|c} 1 & 0 & 3 & 1 & -1 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\text{Answer D: } \begin{pmatrix} 5 \\ -1 \\ 0 \\ 0 \\ 3 \end{pmatrix} + k \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + l \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad k, l \in \mathbb{R} \text{ or equivalently } \begin{pmatrix} 5 \\ -1 \\ 0 \\ 0 \\ 3 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

2. Use the supplied space and include clear argumentation and calculation.

$$\text{Let } A = \begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix}.$$

a) (2 pts.) Find  $A^{-1}$ , the inverse of  $A$ .

b) (2 pts.) By using  $A^{-1}$ , solve the system  $A\mathbf{x} = \mathbf{b}$  where  $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ , that is, find  $\mathbf{x}$ .

Solution:

$$\begin{aligned} \text{a) } (A | I) &= \left( \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array} \right) \sim \left( \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{array} \right) \sim \left( \begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 2 & -2 & 1 \end{array} \right) \sim \left( \begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & 1/2 \end{array} \right) \\ &\Rightarrow A^{-1} = \begin{pmatrix} 3 & -1 \\ -1 & 1/2 \end{pmatrix} \end{aligned}$$

$$\text{b) } A\mathbf{x} = \mathbf{b} \Rightarrow A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 3 & -1 \\ -1 & 1/2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$$

3. (2 pts) Fill in your final answer to this exercise on the supplied answer sheet.

If  $A, B$ , and  $C$  are invertible matrices of size  $n \times n$  each, solve the equation  $C^{-1}(A + X)B^{-1} = I_n$  for  $X$ .

Solution:

Here we need to find the  $X$ . We have the equation:  $C^{-1}(A + X)B^{-1} = I_n$

We can left-multiply the equation by  $C$ :  $CC^{-1}(A + X)B^{-1} = CI_n \Rightarrow I_n(A + X)B^{-1} = C$

Now we can right-multiply the equation by  $B$ :  $I_n(A + X)B^{-1}B = CB \Rightarrow (A + X)I_n = CB$

We now expand the left side and subtract  $A$  from both sides:  $A + X = CB \Rightarrow A + X = CB \Rightarrow X = CB - A$ .

4. (2pts) Use the supplied space and include clear argumentation and calculation.

Let  $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ ,  $\mathbf{v} = \begin{pmatrix} 5 \\ -13 \\ -3 \end{pmatrix}$ , and  $\mathbf{b} = \begin{pmatrix} -3 \\ 8 \\ 1 \end{pmatrix}$ . Then  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$  is a plane through the origin in  $\mathbb{R}^3$ . Is  $\mathbf{b}$  in the mentioned plane? Motivate your answer.

Solution:

If  $\mathbf{b}$  is in the plane then, the equation  $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{b}$  has a solution. To answer this, we row reduce the augmented matrix  $(\mathbf{u} \ \mathbf{v} \mid \mathbf{b})$ :

$$\left( \begin{array}{cc|c} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{array} \right) \sim \left( \begin{array}{cc|c} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{array} \right) \sim \left( \begin{array}{cc|c} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{array} \right)$$

The third equation is  $0 = -2$ , which shows that the system has no solution. The vector equation  $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{b}$  has no solution, so  $\mathbf{b}$  is not in  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$  and hence  $\mathbf{b}$  is not in the plane.

5. Use the supplied space and include clear argumentation and calculation.

Let  $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$ .

a) (3 pts) Determine Null  $A$ .

b) (2 pts) Find a basis for Null  $A$ .

c) (2 pts) Determine the coordinates of  $\begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$  with respect to the basis for Col $A$  given by

$$\mathcal{B}_{\text{Col } A} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Solution:

a)  $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ . Solving the system

for the reduced echelon form we get:

$$\begin{array}{l} x_1 = -x_3 \\ x_2 = x_3 \\ x_3 = x_3 \\ x_4 = 0 \end{array}, \mathbf{x} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, x_3 \in \mathbb{R}.$$

Therefore Null  $A = \text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$  or Null  $A = \text{Span} \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \right\}$  or equivalent.

b) A basis for Null  $A$  is  $\mathcal{B}_{\text{Null } A} = \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \right\}$ .

$$c) [\mathbf{x}]_{\mathcal{B}_{\text{Col } A}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \text{ where } \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$$

$$\Rightarrow \alpha_1 = 2, \alpha_2 = 3, \alpha_3 = -4$$

$$[\mathbf{x}]_{\mathcal{B}_{\text{Col } A}} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

6. (4 pts.) Use the supplied space and include clear argumentation and calculation.

Let  $B \in \mathbb{R}^{n \times n}$ . Given  $B^T \cdot B = I_n$ , show that either  $\det B = 1$  or  $\det B = -1$ .

Solution:

$$\det B^T = \det B,$$

$$\det B^T \cdot B = \det B^T \cdot \det B$$

$$\det I_n = 1$$

$$\text{If } B^T B = I_n \text{ is given then, } \det B^T \cdot B = \det I_n = 1$$

$$\text{From the above, we have } (\det B)^2 = \det I = 1.$$

This implies that  $\det B = 1$  or  $\det B = -1$ .

7. Use the supplied space and include clear argumentation and calculation.

$$\text{For the given matrix } A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

a) (2 pts.) Find all eigenvalues of  $A$ .

b) (2 pts.) Find the corresponding eigenspaces of  $A$ .

Solution:

$$a) A - \lambda I = \begin{pmatrix} 1 - \lambda & -2 \\ 1 & 3 - \lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (1 - \lambda)(3 - \lambda) - (-2) = \lambda^2 - 4\lambda + 5$$

$$\text{The eigenvalues are } \lambda_{1,2} = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \mp i$$

$$\lambda_1 = 2 + i \text{ and } \lambda_2 = 2 - i$$

b) For  $\lambda_1 = 2 + i$

$$A - (2 + i)I = \begin{pmatrix} -1 - i & -2 \\ 1 & 1 - i \end{pmatrix} \sim \begin{pmatrix} 1 & 1 - i \\ 0 & 0 \end{pmatrix}$$

The equation  $(A - \lambda_1 I)\mathbf{v} = \mathbf{0}$  gives

$$\begin{cases} (-1 - i)x_1 - 2x_2 = 0 \\ x_1 + (1 - i)x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -(1 - i)x_2 \\ x_2 = x_2 \end{cases}$$

$$\text{therefore, } E_{2+i} = \text{Span} \left\{ \begin{pmatrix} -1 + i \\ 1 \end{pmatrix} \right\} \text{ and } E_{2-i} = \text{Span} \left\{ \begin{pmatrix} -1 - i \\ 1 \end{pmatrix} \right\}$$

If no row of zeroes would result, 0.5pt might be awarded (once only) for subtracting the (supposed) eigenvalues down the main diagonal.

8. (4 pts.) *Fill in your final answer to this exercise on the supplied answer sheet. Each error results in a one-point deduction from the total score (4 pts).*

Determine whether each of the following statements is true or false.

- i. If  $\mathbf{v}$  is an eigenvector of  $A$ , then  $\mathbf{v}$  is also an eigenvector of  $A^2$ .
- ii. For an invertible matrix  $A$ , if  $\lambda$  is an eigenvalue of  $A$ , then  $\lambda$  is also an eigenvalue of  $A^{-1}$ .
- iii. For an invertible matrix  $A$ , if  $\mathbf{v}$  is an eigenvector of  $A$ , then  $\mathbf{v}$  is also an eigenvector of  $A^{-1}$ .
- iv. If  $\mathbf{v}$  is an eigenvector of  $A$ , then  $\mathbf{v}$  is also an eigenvector of  $A^T$ .
- v. If  $\lambda \neq 0$  is an eigenvalue of  $AB$  (for square matrices  $A$  and  $B$ ) then  $\lambda$  is also an eigenvalue of  $BA$ .

Answer:

Version A:

(i) T, (ii) F, (iii) T, (iv) F, (v) T

Version B:

(i) T, (ii) T, (iii) T, (iv) F, (v) F

Version C:

(i) F, (ii) F, (iii) T, (iv) T, (v) T

Version D:

(i) F, (ii) T, (iii) F, (iv) T, (v) T

9. *Use the supplied space and include clear argumentation and calculation.*

Let  $G$  and  $F$  be linear transformations from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . The linear transformation  $F$  rotates the vectors around the origin over an angle of  $\frac{\pi}{2}$  radians. The representation matrix of  $G \circ F$  is given by  $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ .

- a) (2 pts.) What is the representation matrix of  $F$ ?
- b) (2 pts.) What is the representation matrix of  $G$ ?
- c) (3 pts.) Is  $G \circ F$  one-to-one? Is  $G \circ F$  onto? Motivate your answers.

Solution:

a) The representation matrix of  $F$  is  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .

b) We know that the composition of linear functions can be stated as:

$$BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

where  $A$  and  $B$  are the representation matrices of  $F$  and  $G$  respectively.

Solving the equations we find  $a = -3, b = 1, c = -4, d = 2$ , therefore, the representation matrix of  $G$  is  $B = \begin{pmatrix} -3 & 1 \\ -4 & 2 \end{pmatrix}$ .

c)  $G \circ F$  has the representation matrix as  $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It can be seen that  $\text{Null } BA = \{\mathbf{0}\}$  therefore  $G \circ F$  is one-to-one. Moreover,  $BA$  is invertible, therefore  $G \circ F$  is onto.