

1. Determine the values $x, y, z \in \mathbb{R}$ for which $A = A^{-1}$
If $A = A^{-1}$ then, we have $AA = A^{-1}A = I$.

$$\begin{pmatrix} x & y & 0 \\ 3 & 5 & 0 \\ 1 & z & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The above matrix multiplication leads to: $x = -5, y = -8, z = 2$.

2. Decide, for each transformation given, whether it is one-to-one and whether it is onto.

S_1	:	one-to-one	(×)	onto	(×)
S_2	:	one-to-one	(×)	onto	(✓)
S_3	:	one-to-one	(✓)	onto	(×)
S_4	:	one-to-one	(✓)	onto	(✓)

3. Decide whether the transformation is linear or non-linear, and write the representation matrix for the linear transformation(s):

For all versions:

T_1	:	linear	(✓)	non-linear	(×)
T_2	:	linear	(×)	non-linear	(✓)
T_3	:	linear	(✓)	non-linear	(×)
Representation matrix of T_1 (if linear): $A_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ or $A_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$					
Representation matrix of T_2 (if linear): not linear					
Representation matrix of T_3 (if linear): $A_3 = \begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix}$ or $A_3 = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}$					

4. All eigenvalues for B^5 and respective eigenspaces for each eigenvalue:

The eigenvalues for B^5 are $1^5=1, 3^5=243$, and $5^5=3125$.

The respective eigenspaces are

$$E_{1^5} = \text{Span} \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}, \quad E_{3^5} = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}, \quad E_{5^5} = \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

5. The intersection of the planes p_1 and p_2 in **parametric vector form**:

Rewrite p_2 as:

$$p_2: \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R} \text{ and solve with } p_1.$$

$$p_2: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2\alpha + \beta \\ 1 + \alpha + 2\beta \\ 1 + \alpha + 3\beta \end{pmatrix}$$

Substitute this in p_1 ,

$$(-2\alpha + \beta) + 3(1 + \alpha + 2\beta) - 2(1 + \alpha + 3\beta) = 0$$

$$\beta = \alpha - 3$$

$$\text{Substitute this in } p_2: \text{ We have the intersection as: } \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + (\alpha - 3) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

The intersection is a line and can be written as the following parametric vector form:

$$\begin{pmatrix} -3 \\ -5 \\ -8 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \right\}$$

6. Find $\det C$:

$$\det C = \det[(2B)A^{-1}B^T A] = \det(2B)\det A^{-1}\det B^T \det A$$

$$= 2^3 \det B \frac{1}{\det A} \det B \det A$$

$$= 2^3 (\det B)^2$$

$$B \text{ is upper triangular matrix, hence } \det B = 3 \cdot 1 \cdot (-1) = -3$$

$$\text{Then we have } \det C = 2^3 (\det B)^2 = 8(-3)^2 = 72$$

7. Show that the given set is linearly independent:

Let $c_1, c_2, c_3 \in \mathbb{R}$ satisfy the following equation:

$$c_1(2\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_2 - \mathbf{v}_3) + c_3(\mathbf{v}_1 + \mathbf{v}_2) = 0$$

Then we must show that $c_1 = c_2 = c_3 = 0$.

Rewrite the above equation:

$$(2c_1 + c_3)\mathbf{v}_1 + (c_1 + c_2)\mathbf{v}_2 + (c_3 - c_2)\mathbf{v}_3 = 0$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set, $2c_1 + c_3 = 0$, $c_1 + c_2 = 0$, $c_3 - c_2 = 0$

Determine the echelon form of the above system of linear equations:

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \sim \dots \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow 0.5 \text{ pt}$$

From which, we get $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ as the unique solution.

Hence $\{2\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_3\}$ is a linearly independent set.

8. a) The representation matrix A_1 of T_1 :

$$A_1 = \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}$$

b) The representation matrix A_2 of T_2 :

$$A_2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

c) The representation matrix B of F :

$$B = A_1 \cdot A_2$$

$$B = \begin{pmatrix} 0 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}$$

d) The vectors whose F image is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$:

$$\text{Solve } B \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We get the following augmented matrix: $\left(\begin{array}{cc|c} -\sqrt{2}/2 & -\sqrt{2}/2 & 1 \\ 0 & 0 & 1 \end{array} \right)$, which is inconsistent. Hence

there is no $\mathbf{x}$ whose F image is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

If $B = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$ is taken, $\mathbf{x} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

$$\mathbf{x} \in \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \right\}$$

9. Determine the eigenvalues of M together with the corresponding eigenspaces.

To determine the eigenvalues of M :

$$\det(M - \lambda I) = 0 \rightarrow$$

$$\begin{aligned} \det \begin{pmatrix} -3-\lambda & 0 & -1 \\ 1 & 1-\lambda & 2 \\ 5 & 0 & 1-\lambda \end{pmatrix} &= (1-\lambda) \det \begin{pmatrix} -3-\lambda & -1 \\ 5 & 1-\lambda \end{pmatrix} = \\ &= (1-\lambda)[(-3-\lambda)(1-\lambda)+5] \\ &= (1-\lambda)(\lambda^2+2\lambda+2) \\ &= (1-\lambda)(\lambda+1-i)(\lambda+1+i) \end{aligned}$$

Hence the eigenvalues are: $1, -1+i, -1-i$

To find E_1 , reduce the matrix $A - I = \begin{pmatrix} -4 & 0 & -1 \\ 1 & 0 & 2 \\ 5 & 0 & 0 \end{pmatrix} \sim \dots \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$$E_1 = \text{Null}(A - I) = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

To find E_{-1+i} , reduce the matrix $A - (-1+i)I = \begin{pmatrix} -2-i & 0 & -1 \\ 1 & 2-i & 2 \\ 5 & 0 & 2-i \end{pmatrix} \sim \dots \sim \begin{pmatrix} 1 & 0 & (2-i)/5 \\ 0 & 1 & (3+i)/5 \\ 0 & 0 & 0 \end{pmatrix}$

$$E_{-1+i} = \text{Null}(A - (-1+i)I) = \text{Span} \left\{ \begin{pmatrix} -2+i \\ -3-2i \\ 5 \end{pmatrix} \right\}$$

To find E_{-1-i} take the conjugate of the vector that spans E_{-1+i}

$$E_{-1-i} = \text{Null}(A - (-1-i)I) = \text{Span} \left\{ \begin{pmatrix} -2-i \\ -3+2i \\ 5 \end{pmatrix} \right\}$$