

Q1. (a) $r(t) = x(t)i + y(t)j$ with $x(t) = 2\cos(t) + 1$, $y(t) = \sin(t)$, $0 \leq t \leq 2\pi$

(b) $\text{mass} = \int_C \rho(x,y) ds$

$$= \int_0^{2\pi} \rho(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

$$= \int_0^{2\pi} \sqrt{\cos^2(t) + 4\sin^2(t)} \sqrt{4\sin^2(t) + \cos^2(t)} dt$$

$$= \int_0^{2\pi} 1 + 3\sin^2(t) dt$$

$$= 2\pi + \frac{3}{2} \left[t - \sin(t)\cos(t) \right]_0^{2\pi}$$

$$= 5\pi$$

Q.2. (a)

$$\text{flux} = \int_C \mathbf{F} \cdot \mathbf{n} \, ds$$

$$= \int_{C_1} \mathbf{F} \cdot \mathbf{n} \, ds + \int_{C_2} \mathbf{F} \cdot \mathbf{n} \, ds$$

ad C_1 : $\mathbf{r}_1(t) = \underbrace{(1-t)}_{x(t)} \mathbf{i}, 0 \leq t \leq 2$

$M dy = 0$ and $N dx = 0$.

$$\Rightarrow \int_{C_1} \mathbf{F} \cdot \mathbf{n} \, ds = \int_{C_1} M dy - N dx = 0$$

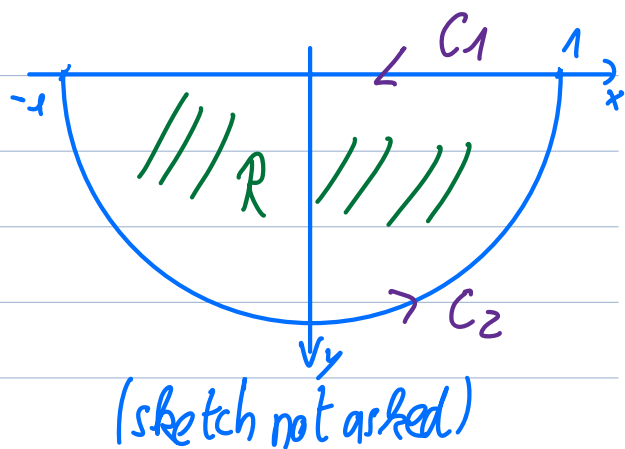
ad C_2 : $\mathbf{r}_2(t) = (\cos t) \mathbf{i} + (\sin t) \mathbf{j} = x(t) \mathbf{i} + y(t) \mathbf{j}, \pi \leq t \leq 2\pi$

$$M dy = x(t)^2 y'(t) dt = \cos(t)^3 dt$$

$$N dx = x(t) y(t) x'(t) dt = -\cos(t) \sin(t)^2 dt$$

$$\begin{aligned} \Rightarrow \int_{C_2} \mathbf{F} \cdot \mathbf{n} \, ds &= \int_{C_2} M dy - N dx = \int_{\pi}^{2\pi} \underbrace{\cos(t)^3 + \cos(t) \sin(t)^2}_{\substack{= \cos(t) (\cos^2(t) + \sin^2(t)) \\ = \cos(t)}} dt \\ &= 0 \end{aligned}$$

$$\Rightarrow \text{flux} = 0$$



(b) $\text{flux} = \int_C \mathbf{F} \cdot \mathbf{n} \, ds \stackrel{\text{Green}}{=} \iint_R \text{div}(\mathbf{F}) \, dA$

$$= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 3x \, dy \, dx$$

$$= 3 \int_{-1}^0 x \sqrt{1-x^2} \, dx + 3 \int_0^1 x \sqrt{1-x^2} \, dx$$

$$= -3 \int_0^1 x \sqrt{1-x^2} \, dx + 3 \int_0^1 x \sqrt{1-x^2} \, dx = 0$$

Q.3(a) F is conservative $\Leftrightarrow \nabla \times F = 0$

$$\nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ ay + 2xe^{x^2-z^2} & bx & z - 2ze^{x^2-z^2} \end{vmatrix}$$

$$= (0-0)\mathbf{i} - (-4xz e^{x^2-z^2} + 4xz e^{x^2-z^2})\mathbf{j} + (b-a)\mathbf{k}$$

$$= (b-a)\mathbf{k}$$

Thus F is conservative, iff $a=b$.

(b) Perform subsequent integration:

$$\frac{\partial f}{\partial x} = y + 2xe^{x^2-z^2} \Rightarrow f(x,y,z) = xy + e^{x^2-z^2} + g(y,z)$$

$$\frac{\partial f}{\partial y} = x \Rightarrow x + \frac{\partial g}{\partial y} = x \Rightarrow g(y,z) = h(z)$$

$$\frac{\partial f}{\partial z} = z - 2ze^{x^2-z^2} \Rightarrow -2ze^{x^2-z^2} + h'(z) = z - 2ze^{x^2-z^2} \Rightarrow h(z) = \frac{1}{2}z^2 + C, C \in \mathbb{R}$$

$\Rightarrow$ claim: $f(x,y,z) = xy + e^{x^2-z^2} + \frac{1}{2}z^2 + C, C \in \mathbb{R}$ is a potential for F .

verification: we compute $\nabla f = F$.

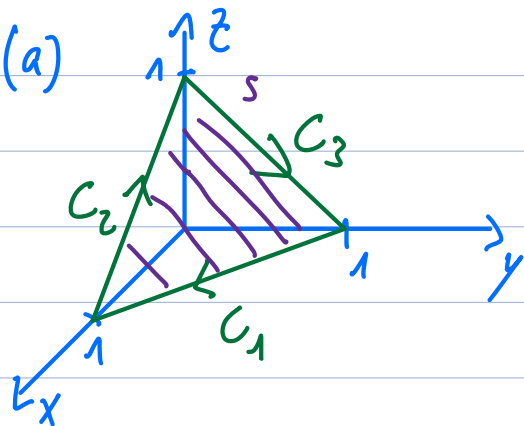
$$(c) \text{ work} = \int_A^B \mathbf{F} \cdot \mathbf{T} ds$$

fund.

$$_{\text{Thm}} = f(B) - f(A)$$

$$= 4 + 1 + 8 - (0 + 1 + 2) = 10$$

Q4.(a)



$$C = C_1 \oplus C_2 \oplus C_3$$

(b) $S: r(x,y) = xi + yj + (1-x-y)k, 0 \leq x \leq 1, 0 \leq y \leq 1-x$

(c) circulation = $\oint_C F \cdot T dr$

Stokes

$$= \iint_S \nabla \times F \cdot n d\sigma$$

$$= \pm \int_0^1 \int_0^{1-x} (\nabla \times F) \cdot (r_x \times r_y) dy dx$$

We have

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = i + j + k$$

$$\nabla \times F = -i - j - k$$

$$= - \int_0^1 \int_0^{1-x} (-3) dy dx$$

n has negative

k-component (RHR)

$$= \int_0^1 3(1-x) dx = \frac{3}{2}$$