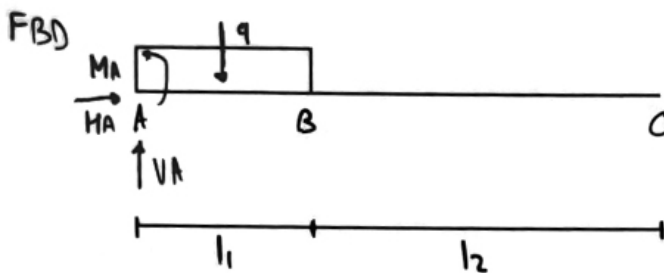
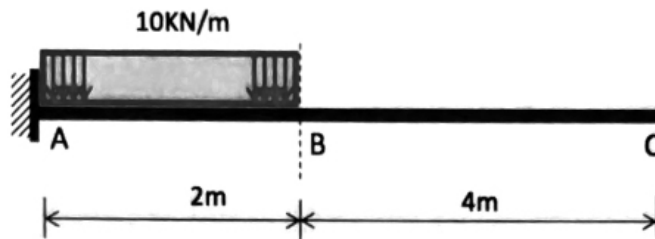


Exercise 1 (30 pt)

The cantilever beam in the figure is supported by a fixed support in A. A distributed load is applied from A to B. Dimensions are reported in the figure. EI is constant.

- Derive the equation of shear force, bending moment, rotation, vertical displacement. In your calculation state explicitly the boundary conditions to apply.
- Sketch the diagrams of shear force, bending moment and displacement.
- Calculate the rotation the vertical displacement in B and C, in terms of EI .



$$\sum F_v = 0 \rightarrow V_A - q \cdot l_1 = 0 \rightarrow V_A = q l_1 \rightarrow V_A = 20 \text{ kN}.$$

$$\sum M_A = 0 \rightarrow M_A - q \frac{l_1^2}{2} = 0 \rightarrow M_A = 20 \text{ kNm}$$

Equations

(30pt)

$$0 \leq x < l_1 \quad \text{I}$$

$$\sum F_v = 0 \rightarrow -V_A + V_x + q x = 0 \rightarrow V(x) = V_A - q x$$

$$\sum M^{(I)} = 0 \rightarrow M_x + M_A - V_A x + q \frac{x^2}{2} \rightarrow M(x) = -M_A + V_A x - q \frac{x^2}{2} \quad 4$$

$$\theta^{(I)} = \frac{1}{EI} \int M(x)^{(I)} dx = \frac{1}{EI} \left[-M_A x + \frac{V_A x^2}{2} - q \frac{x^3}{6} \right] + C_1$$

$$v^{(I)} = \frac{1}{EI} \int \theta(x)^{(I)} dx = \frac{1}{EI} \left[-\frac{M_A x^2}{2} + \frac{V_A x^3}{6} - q \frac{x^4}{24} \right] + C_1 x + C_2. \quad 2$$

$$l_1 < x \leq l_1 + l_2 \quad \text{II}$$

$$\sum F_v = 0 \rightarrow -V_A + q l_1 + V_x \rightarrow V(x) = V_A - q l_1$$

$$\sum M^{(II)} = 0 \rightarrow M_x = 0 \rightarrow M_x = V_A x - q l_1 x \rightarrow M_x = (0) x \rightarrow M_x = 0$$

$$\theta^{(IV)} = \frac{1}{EI} \int M(x)^{(III)} dx = \frac{1}{EI} [0] + C_3$$

$$V^{(IV)} = \frac{1}{EI} \int \theta(x)^{(III)} dx = \frac{1}{EI} [C_3 x] + C_4$$

Boundary conditions

$$\left. \begin{aligned} \theta^{(I)}(x=0) &= 0 \rightarrow C_1 = 0 \\ V^{(I)}(x=0) &= 0 \rightarrow C_2 = 0 \end{aligned} \right\} \text{ supports}$$

(5pt)

$$\left. \begin{aligned} \theta^{(II)}(x=l_1) &= \theta^{(I)}(x=l_1) \\ V^{(II)}(x=l_1) &= V^{(I)}(x=l_1) \end{aligned} \right\} \text{ continuity}$$

$$\frac{1}{EI} \left[-MA l_1 + \frac{VA l_1^2}{2} - q \frac{l_1^3}{6} \right] = C_3 \rightarrow C_3 = \frac{1}{EI} \left[-20 \cdot 2 + \frac{20 \cdot 4}{2} - \frac{10 \cdot 8}{6} \right] - C_3 = \frac{1}{EI} \left[-\frac{40}{3} \right]$$

$$\frac{1}{EI} \left[-\frac{MA l_1^2}{2} + \frac{VA l_1^3}{6} - q \frac{l_1^4}{24} \right] = C_3 l_1 + C_4 \rightarrow C_4 = \frac{1}{EI} \left[-\frac{20 \cdot 4}{2} + \frac{20 \cdot 8}{6} - \frac{10 \cdot 16}{24} + \frac{40 \cdot 2}{3} \right]$$

$$\rightarrow C_4 = \frac{1}{EI} \left[\frac{20}{3} \right]$$

Final equations:

$$\theta^{(I)} = \frac{1}{EI} \left[-MAx + \frac{VAx^2}{2} - q \frac{x^3}{6} \right]$$

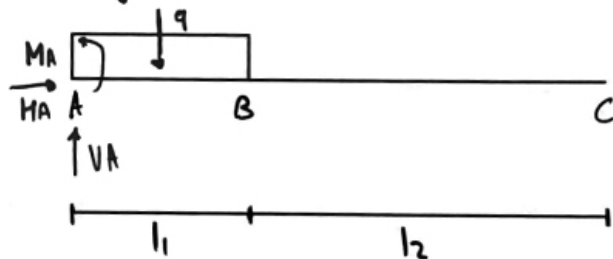
$$V^{(I)} = \frac{1}{EI} \left[-\frac{MAx^2}{2} + \frac{VAx^3}{6} - q \frac{x^4}{24} \right]$$

$$\theta^{(IV)} = \frac{1}{EI} \left[-\frac{40}{3} \right]$$

$$V^{(IV)} = \frac{1}{EI} \left[-\frac{40}{3} x + \frac{20}{3} \right]$$

(5pt)

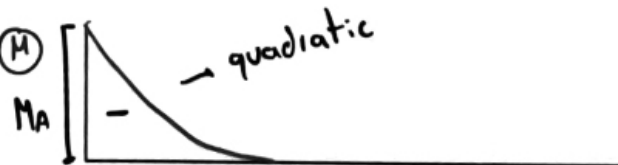
Diagrams



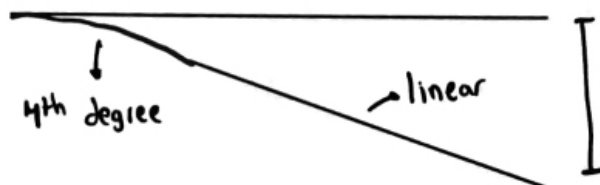
(V)



(M)



(V)



$$= \frac{73.3}{EI}$$

2

Rotation and vertical displacement.

5 pt

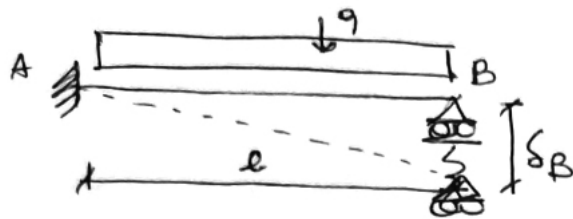
$$\textcircled{B} \quad \theta^{(II)}(l_1) = -\frac{13.33}{EI}$$

$$v^{(II)}(l_1) = \frac{1}{EI} \left[-\frac{40.2}{3} + \frac{20}{3} \right] = -\frac{20}{EI}$$

$$\textcircled{C} \quad \theta^{(II)}(l_1 + l_2) = -\frac{13.33}{EI}$$

$$v^{(II)}(l_1 + l_2) = \frac{1}{EI} \left[-\frac{40.6}{3} + \frac{20}{3} \right] = -\frac{73.3}{EI}$$

EXERCISE 2 [30 pt]



$$q = 10 \text{ kN/m}$$

$$l = 6 \text{ m}$$

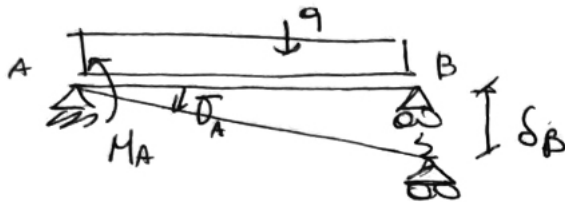
$$\delta_B = 5 \text{ mm} \text{ settlement}$$

$$E = 200 \text{ GPa} = 200 \times 10^6 \text{ kN/m}^2$$

$$I = 800 \times 10^6 \text{ mm}^4 = 0.3 \times 10^{-3} \text{ m}^4$$

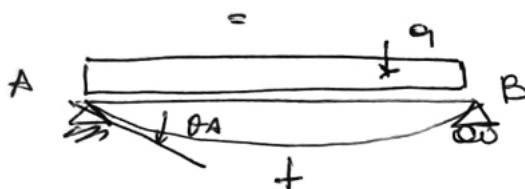
Equivalent statically determined beam:

(10 pt)



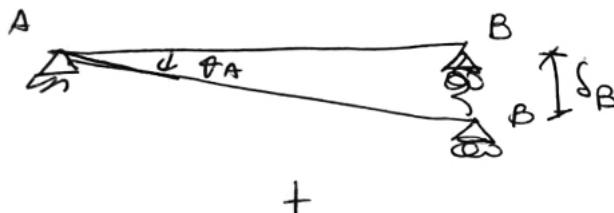
compatibility equation:

$$\theta_A(q, M_A, \delta_B) = 0$$



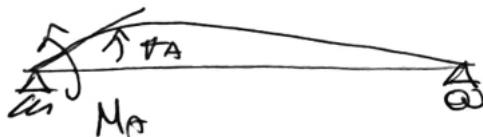
$$\theta_A(q) = \frac{q l^3}{24 EI}$$

2



$$\theta_A(\delta_B) = -\frac{\delta_B}{l}$$

5



$$\theta_A(M_A) = \frac{1}{3} \frac{M l}{EI}$$

3

compatibility: $-\frac{q l^3}{24 EI} + \frac{1}{3} \frac{M l}{EI} - \frac{\delta_B}{l} = 0$

(5 pt)

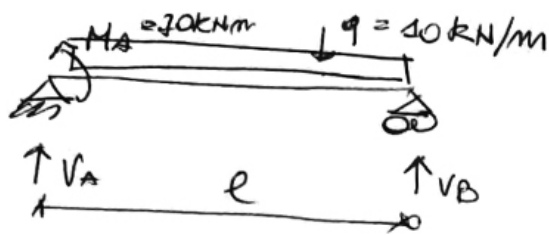
$$-\frac{10 \times 6^3}{24 \times 200 \times 10^6 \times 0.3 \times 10^{-3}} + \frac{1}{3} \frac{M_A \times 6}{200 \times 10^6 \times 0.3 \times 10^{-3}} - \frac{5 \times 10^{-3}}{6} = 0$$

$$\rightarrow M_A = \underline{70 \text{ kNm}}$$

(2 pt)

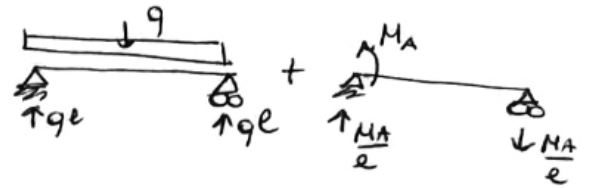
I can solve the beam:

other reactions



8pt

procedure 4

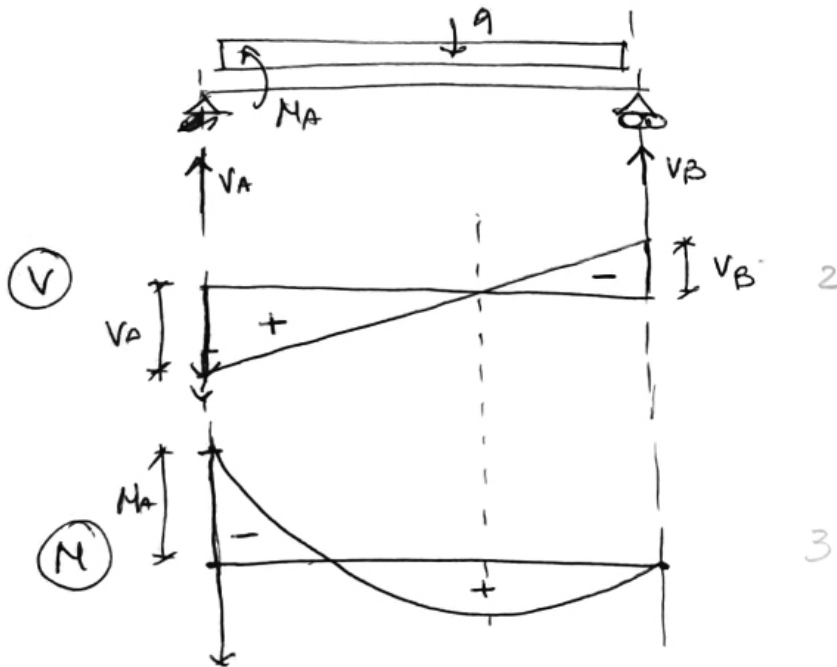


using superposition of effects:

$$V_A = \frac{ql}{2} + \frac{M_A}{l} = \frac{10 \times 6}{2} + \frac{70}{6} = 30 + 11,67 = 41,67 \text{ kN}$$

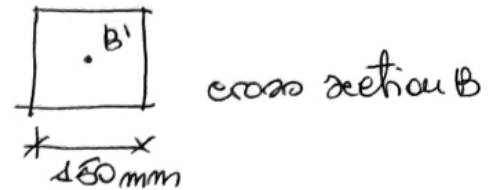
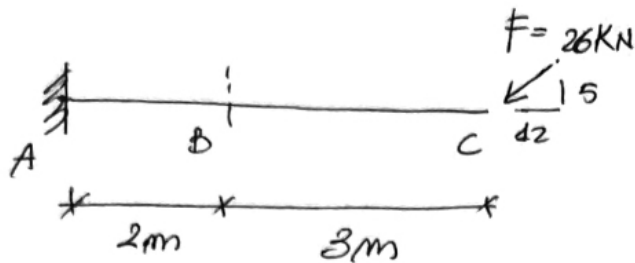
$$V_B = \frac{ql}{2} - \frac{M_A}{l} = \frac{10 \times 6}{2} - \frac{70}{6} = 18,33 \text{ kN}$$

Diagrams:



5pt

EXERCISE 3 [40 pt]

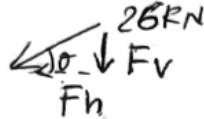


$$\theta = \arctan(5/12) = 22,62^\circ$$

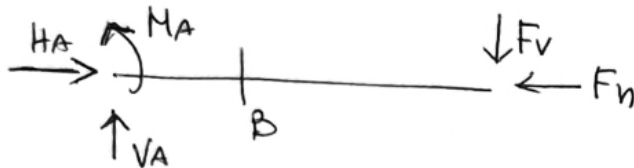
$$F_h = F \cos(\theta) = 24 \text{ kN}$$

$$F_v = F \sin(\theta) = 10 \text{ kN}$$

Forces in C :



Solving the beam (not strictly needed):



$$H_A = F_h = 24 \text{ kN}$$

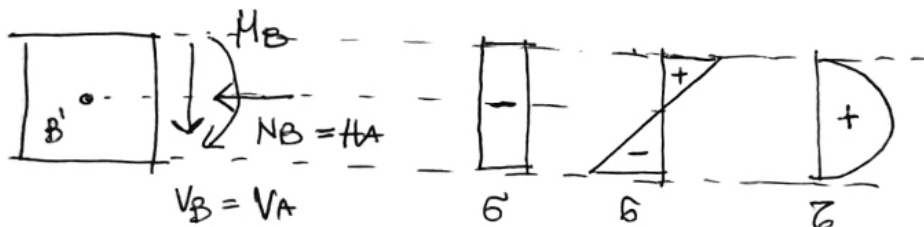
$$V_A = F_v = 10 \text{ kN}$$

$$M_A = F_v l =$$

$$= 10 \times (2+3) = 50 \text{ kNm}$$

State of stress in B :

①



5pt

If needed M_B can be calculated:

$$M_B = -M_A + V_A \times 2 \text{ m} = -50 \text{ kNm} + 10 \times 2 \text{ m} = -30 \text{ kNm}$$

$$N_B = H_A = 24 \text{ kN} \quad \text{compression} < 0$$

$$V_B = V_A = +10 \text{ kN} > 0$$

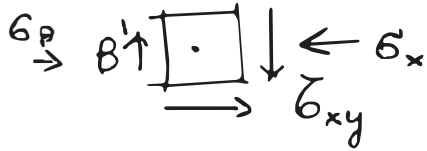
in B' :

$$\sigma_{B'} = \frac{N_B}{A} \pm \frac{M_B c}{I} = \frac{-24 \text{ kN}}{150^2 \text{ mm}^2} \approx -1 \text{ MPa}$$

10pt

$$\tau_{B'} = 1,5 \frac{V_B}{A} = 1,5 \frac{10 \text{ kN}}{150^2 \text{ mm}^2} \approx 0,67 \text{ MPa}$$

Stress representation in B' :



5pt

because of the frame of reference
on the right vertical face

$$\sigma_x < 0 = -1 \text{ MPa}$$

$$\sigma_{xy} < 0 = -0,67 \text{ MPa}$$

$$\sigma_y = 0$$

(2)

10 pt $\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{-1}{2} = -0,5 \text{ MPa}$ 2

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2} = \sqrt{\left(\frac{-1}{2}\right)^2 + (-0,67)^2} = 0,836 \text{ MPa}$$

2

$$\sigma_1 = \sigma_{avg} + R = 0,34 \text{ MPa}$$

$$\sigma_2 = \sigma_{avg} - R = -1,34 \text{ MPa}$$

2

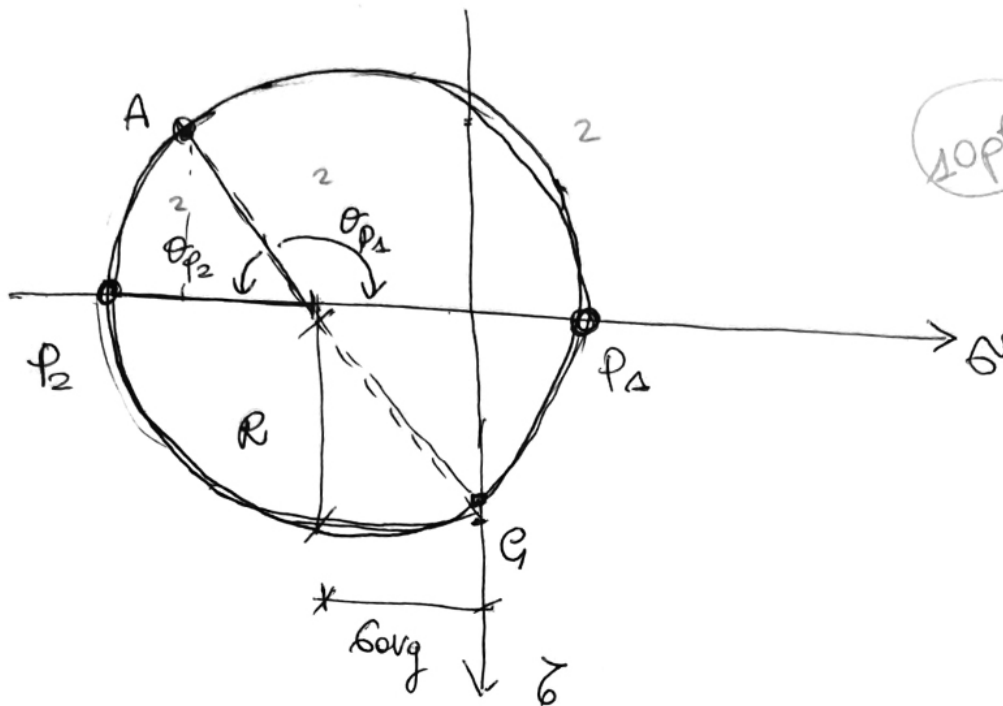
$$\alpha \quad \theta_p = \frac{1}{2} \arctan \left(\frac{\sigma_{xy}}{(\sigma_x - \sigma_y)/2} \right) = \frac{1}{2} \arctan \left(\frac{-0,67}{-1/2} \right) = \frac{26,6^\circ}{2}$$

$$\theta_p = \theta_p + 90^\circ = 116,6^\circ \quad (\text{or } -63,4^\circ)$$

③ Mohr circle

$$A = (\sigma_x; \tau_{xy}) = (-1; -0,67)$$

$$B = (\sigma_y; -\tau_{xy}) = (0; +0,67)$$



The values showed in the circle are consistent
to those calculated analytically.