

Final answer question:

Q1:

	version A	version B	version C	version D
(a)	F	F	T	F
(b)	T	F	F	F
(c)	F	T	F	F
(d)	F	F	F	T

Q2:

version A	version B	version C	version D
$\begin{pmatrix} -9 \\ -5 \\ -9 \end{pmatrix}$	$\begin{pmatrix} -7 \\ -4 \\ -7 \end{pmatrix}$	$\begin{pmatrix} -5 \\ -3 \\ -5 \end{pmatrix}$	$\begin{pmatrix} -3 \\ -2 \\ -3 \end{pmatrix}$

Q3:

	version A	version B	version C	version D
(a)	$C \neq -1$	$C \neq -2$	$C \neq -3$	$C \neq -4$
(b)	$\begin{pmatrix} -\frac{1}{6} & 1 & -1 \\ -\frac{1}{6} & 0 & 1 \\ \frac{1}{6} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{7} & 1 & -1 \\ -\frac{1}{7} & 0 & 1 \\ \frac{1}{7} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{8} & 1 & -1 \\ -\frac{1}{8} & 0 & 1 \\ \frac{1}{8} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{9} & 1 & -1 \\ -\frac{1}{9} & 0 & 1 \\ \frac{1}{9} & 0 & 0 \end{pmatrix}$
(c)	$C \neq -1$	$C \neq -2$	$C \neq -3$	$C \neq -4$

Q4:

	version A	version B	version C	version D
	$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

Question 5. Write a full calculation/argumentation clearly in the box.

We know that $\begin{pmatrix} 1 \\ y \end{pmatrix}$ belongs to $\text{col}(B)$, therefore

$\begin{pmatrix} 1 \\ y \end{pmatrix}$ belongs to $\text{Null}(A)$. Thus $A \begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Therefore

$$\begin{pmatrix} x & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x - y = 0 \\ 2 - 2y = 0 \Rightarrow 2 = 2y \\ \Rightarrow y = 1 \end{cases}$$

and $x = y = 1$.

We also know that $\begin{pmatrix} z \\ 1 \end{pmatrix}$ belongs to $\text{col}(B)$

Therefore, $\begin{pmatrix} z \\ 1 \end{pmatrix}$ belongs to $\text{Null}(A)$

i.e. $A \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Thus

$$\begin{pmatrix} x & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} xz - 1 = 0 \\ 2z - 2 = 0 \Rightarrow z = 1 \end{cases}$$

Hence we have that:

$$x = 1, y = 1, \text{ and } z = 1.$$

Question 6a. Write a full calculation/argumentation clearly in the box.

We have:

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & -2 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & 2 & 0 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 / -2}$$

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_3} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_2}$$

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 / 2} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_3}$$

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -1 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_3}$$

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{Therefore a basis for } H \text{ is } \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Question 6b. Write a full calculation/argumentation clearly in the box.

Since H is a subspace of $\mathbb{R}^3$ that has dimension 3, then we must have that $H = \mathbb{R}^3$

Question 7a. Write a full calculation/argumentation clearly in the box.

We have that

$$\text{Null}(A) = \text{Null}(A - 0I) = E_0 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Question 7b. Write a full calculation/argumentation clearly in the box.

Since A has three distinct eigenvalues, we know that it is diagonalizable.

Moreover, we know that if $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

and $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

then $A = PDP^{-1}$

Thus

$$A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T = (P^T)^{-1} D^T P^T$$

Thus we can take

$$L = D^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

and

$$Q = P^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Question 8a. Write a full calculation/argumentation clearly in the box.

We know that $A = \begin{pmatrix} 1 & 1 \\ \tau(\vec{e}_1) & \tau(\vec{e}_2) \end{pmatrix}$ and

$$\text{that } \left| \det \begin{pmatrix} 1 & 1 \\ \tau(\vec{e}_1) & \tau(\vec{e}_2) \end{pmatrix} \right| = 2$$

$$\Rightarrow \det(A) \neq 0$$

Therefore A is invertible

Question 8b. Write a full calculation/argumentation clearly in the box.

$$\text{We know that } \left| \det \begin{pmatrix} 2 & 1 \\ 1 & x \end{pmatrix} \right| = 2$$

$$\text{Therefore, } |2x - 1| = 2$$

$$\Rightarrow \begin{array}{l} 2x - 1 = -2 \\ 2x = -1 \end{array} \quad \text{or} \quad \begin{array}{l} 2x - 1 = 2 \\ 2x = 3 \end{array}$$

$$\Rightarrow \begin{array}{l} x = -1/2 \\ \text{or} \\ x = 3/2 \end{array}$$

Question 9a. Write a full calculation/argumentation clearly in the box.

Since $\vec{v}$ is a solution to the system $A\vec{x}=\vec{b}$ we have that $A\vec{v}=\vec{b}$ (*)

Since also $2\vec{v}$ is a solution to $A\vec{x}=\vec{b}$ we have that $A(2\vec{v})=\vec{b}$

On the other hand $A(2\vec{v})=2A\vec{v} \stackrel{(*)}{=} 2\vec{b}$

Thus $2\vec{b}=\vec{b}$

$\Rightarrow \vec{b}=\vec{0}$

Question 9b. Write a full calculation/argumentation clearly in the box.

First, the system $A\vec{x}=\vec{0}$ is ^{always} consistent because $\vec{x}=\vec{0}$ is a solution.

Moreover, since A is of size 3×4 we see that the system $A\vec{x}=\vec{0}$ has 3 equations and 4 unknowns.

Therefore, it must have at least one free variable and since it is consistent it must have infinitely many solutions.

Question 10a. Write a full calculation/argumentation clearly in the box.

We need to show:

a) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for every $\vec{u}, \vec{v}$ in $\mathbb{R}^2$

b) $T(c\vec{u}) = cT(\vec{u})$ for every $c \in \mathbb{R}$ and $\vec{u} \in \mathbb{R}^2$

Proof of a):

$$\begin{aligned} T(\vec{u} + \vec{v}) &= T_1(\vec{u} + \vec{v}) - T_2(T_1(\vec{u} + \vec{v})) = \\ &= T_1(\vec{u}) + T_1(\vec{v}) - T_2(T_1(\vec{u}) + T_1(\vec{v})) = \\ &= T_1(\vec{u}) + T_1(\vec{v}) - T_2(T_1(\vec{u})) - T_2(T_1(\vec{v})) \\ &= T_1(\vec{u}) - T_2(T_1(\vec{u})) + T_1(\vec{v}) - T_2(T_1(\vec{v})) = T(\vec{u}) + T(\vec{v}) \end{aligned}$$

Proof of b):

$$\begin{aligned} T(c\vec{u}) &= T_1(c\vec{u}) - T_2(T_1(c\vec{u})) = cT_1(\vec{u}) - T_2(cT_1(\vec{u})) = \\ &= cT_1(\vec{u}) - cT_2(T_1(\vec{u})) = c(T_1(\vec{u}) - T_2(T_1(\vec{u}))) \\ &= cT(\vec{u}) \end{aligned}$$

This proves that T is a linear transformation.

Question 10b. Write a full calculation/argumentation clearly in the box.

Let $\vec{v} \in \ker(T_1)$. Then $T_1(\vec{v}) = \vec{0}$.

$$\text{Thus } T(\vec{v}) = T_1(\vec{v}) - T_2(T_1(\vec{v})) = \vec{0} - T_2(\vec{0})$$

$$= \vec{0} - \vec{0} = \vec{0} \quad \text{i.e. } \vec{v} \in \ker(T)$$