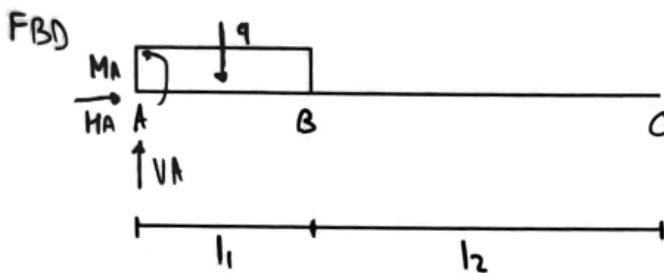
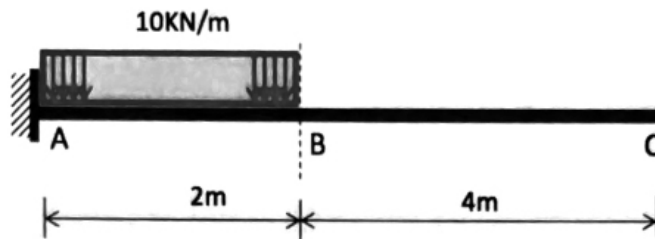


Exercise 1 (30 pt)

The cantilever beam in the figure is supported by a fixed support in A. A distributed load is applied from A to B. Dimensions are reported in the figure. EI is constant.

- Derive the equation of shear force, bending moment, rotation, vertical displacement. In your calculation state explicitly the boundary conditions to apply.
- Sketch the diagrams of shear force, bending moment and displacement.
- Calculate the rotation the vertical displacement in B and C, in terms of EI .



$$\sum F_v = 0 \rightarrow V_A - q \cdot l_1 = 0 \rightarrow V_A = q l_1 \rightarrow V_A = 20 \text{ kN.}$$

$$\sum M_A = 0 \rightarrow M_A - q \frac{l_1^2}{2} = 0 \rightarrow M_A = 20 \text{ kNm}$$

Equations

(30pt)

$$0 \leq x < l_1 \quad \textcircled{\text{I}}$$

$$\sum F_v = 0 \rightarrow -V_A + V_x + q x = 0 \rightarrow V(x) = V_A - q x$$

$$\sum M^{(x)} = 0 \rightarrow M_x + M_A - V_A x + q \frac{x^2}{2} \rightarrow M(x) = -M_A + V_A x - q \frac{x^2}{2} \quad 4$$

$$\theta^{(x)} = \frac{1}{EI} \int M(x)^{(x)} dx = \frac{1}{EI} \left[-M_A x + \frac{V_A x^2}{2} - q \frac{x^3}{6} \right] + C_1$$

$$v^{(x)} = \frac{1}{EI} \int \theta(x)^{(x)} dx = \frac{1}{EI} \left[-\frac{M_A x^2}{2} + \frac{V_A x^3}{6} - q \frac{x^4}{24} \right] + C_1 x + C_2 \quad 2$$

$$l_1 < x \leq l_1 + l_2 \quad \textcircled{\text{II}}$$

$$\sum F_v = 0 \rightarrow -V_A + q l_1 + V_x \rightarrow V(x) = V_A - q l_1$$

$$\sum M^{(x)} = 0 \rightarrow M_x = 0 \rightarrow M_x = V_A x - q l_1 x \rightarrow M_x = (0) x \rightarrow M_x = 0 \quad 2$$

$$\theta^{(IV)} = \frac{1}{EI} \int M(x)^{(III)} dx = \frac{1}{EI} [0] + C_3$$

$$V^{(IV)} = \frac{1}{EI} \int \theta(x)^{(III)} dx = \frac{1}{EI} [C_3 x] + C_4$$

Boundary conditions

$$\left. \begin{aligned} \theta^{(I)}(x=0) &= 0 \rightarrow C_1 = 0 \\ V^{(I)}(x=0) &= 0 \rightarrow C_2 = 0 \end{aligned} \right\} \text{ supports}$$

(5pt)

$$\left. \begin{aligned} \theta^{(II)}(x=l_1) &= \theta^{(I)}(x=l_1) \\ V^{(II)}(x=l_1) &= V^{(I)}(x=l_1) \end{aligned} \right\} \text{ continuity}$$

$$\frac{1}{EI} \left[-MA l_1 + \frac{VA l_1^2}{2} - q \frac{l_1^3}{6} \right] = C_3 \rightarrow C_3 = \frac{1}{EI} \left[-20 \cdot 2 + \frac{20 \cdot 4}{2} - \frac{10 \cdot 8}{6} \right] - C_3 = \frac{1}{EI} \left[-\frac{40}{3} \right]$$

$$\frac{1}{EI} \left[-\frac{MA l_1^2}{2} + \frac{VA l_1^3}{6} - q \frac{l_1^4}{24} \right] = C_3 l_1 + C_4 \rightarrow C_4 = \frac{1}{EI} \left[-\frac{20 \cdot 4}{2} + \frac{20 \cdot 8}{6} - \frac{10 \cdot 16}{24} + \frac{40 \cdot 2}{3} \right]$$

$$\rightarrow C_4 = \frac{1}{EI} \left[\frac{20}{3} \right]$$

Final equations:

$$\theta^{(I)} = \frac{1}{EI} \left[-MAx + \frac{VAx^2}{2} - q \frac{x^3}{6} \right]$$

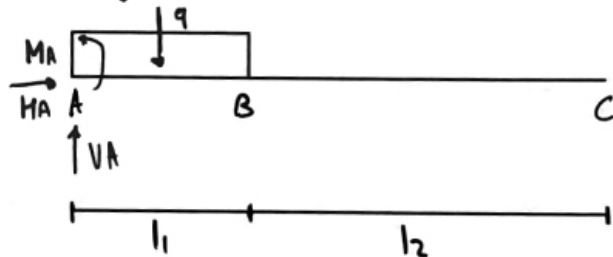
$$V^{(I)} = \frac{1}{EI} \left[-\frac{MAx^2}{2} + \frac{VAx^3}{6} - q \frac{x^4}{24} \right]$$

$$\theta^{(IV)} = \frac{1}{EI} \left[-\frac{40}{3} \right]$$

$$V^{(IV)} = \frac{1}{EI} \left[-\frac{40}{3} x + \frac{20}{3} \right]$$

(5pt)

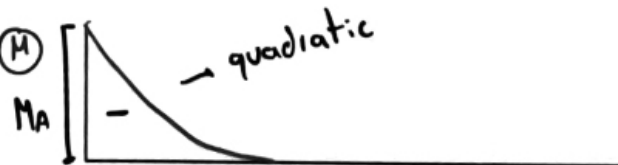
Diagrams



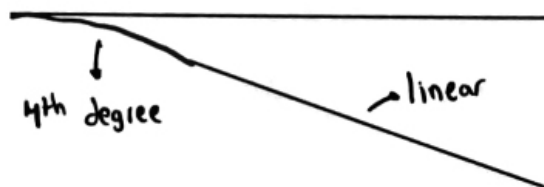
(V)



(M)



(V)



$$= \frac{73.3}{EI}$$

2

Rotation and vertical displacement.

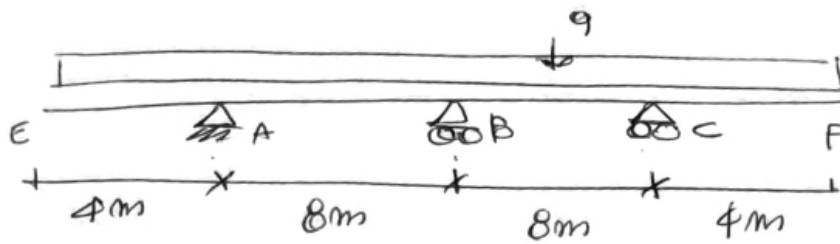
$$\textcircled{B} \quad \theta^{(II)}(l_1) = -\frac{13.33}{EI}$$

$$v^{(II)}(l_1) = \frac{1}{EI} \left[-\frac{40.2}{3} + \frac{20}{3} \right] = -\frac{20}{EI}$$

$$\textcircled{C} \quad \theta^{(II)}(l_1 + l_2) = -\frac{13.33}{EI}$$

$$v^{(II)}(l_1 + l_2) = \frac{1}{EI} \left[-\frac{40.6}{3} + \frac{20}{3} \right] = -\frac{73.3}{EI}$$

EXERCISE 2 [35 pt] - OPTION 1



$$q = 30 \text{ kN/m}$$

$$l_1 = 4 \text{ m}$$

$$l_2 = 8 \text{ m}$$

→ Stiffness factors :

$$(A) \quad K_{AE} = 0 \quad K_{AB} = \frac{4EI}{l_2} = \frac{4EI}{8} = \frac{1}{2} EI$$

$$(B) \quad K_{BA} = \frac{3EI}{l_2} = K_{BC} = \frac{3}{8} EI$$

(5pt)

$$(C) \quad K_{CF} = 0 \quad K_{CB} = \frac{4EI}{l_2} = \frac{1}{2} EI$$

→ Distribution factors :

$$\mu_{AE} = \frac{K_{AE}}{\sum K_A} = 0 \quad \mu_{AB} = \frac{K_{AB}}{\sum K_A} = 1$$

$$\mu_{BA} = \frac{K_{BA}}{\sum K_B} = \frac{1}{2} = \mu_{BC}$$

(2pt)

$$\mu_{CF} = \frac{K_{CF}}{K_C} = 0 \quad \mu_{CB} = \frac{K_{CB}}{\sum K_C} = 1$$

→ Carry-over factors :

$$M_{EA} = 0$$

$$M_{AB} = 0$$

$$M_{CB} = 0$$

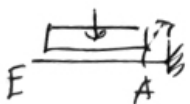
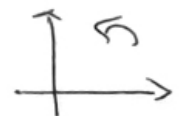
$$M_{FC} = 0$$

$$M_{BA} = \frac{1}{2} M_{AB}$$

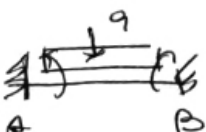
$$M_{BC} = \frac{1}{2} M_{CB}$$

(5pt)

→ Fixed-End moments :



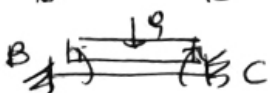
$$M_{AE} = -\frac{1}{2} q l_1^2 = -\frac{1}{2} 30 \times 4^2 = -240 \text{ kNm} \quad M_{EA} = 0$$



$$M_{AB} = +\frac{1}{12} q l_2^2 = \frac{1}{12} 30 \times 8^2 = +160 \text{ kNm}$$

$$M_{BA} = -160 \text{ kNm}$$

(6pt)



$$M_{BC} = +160 \text{ kNm}$$

$$M_{CB} = -160 \text{ kNm}$$



$$M_{CF} = +240 \text{ kNm}$$

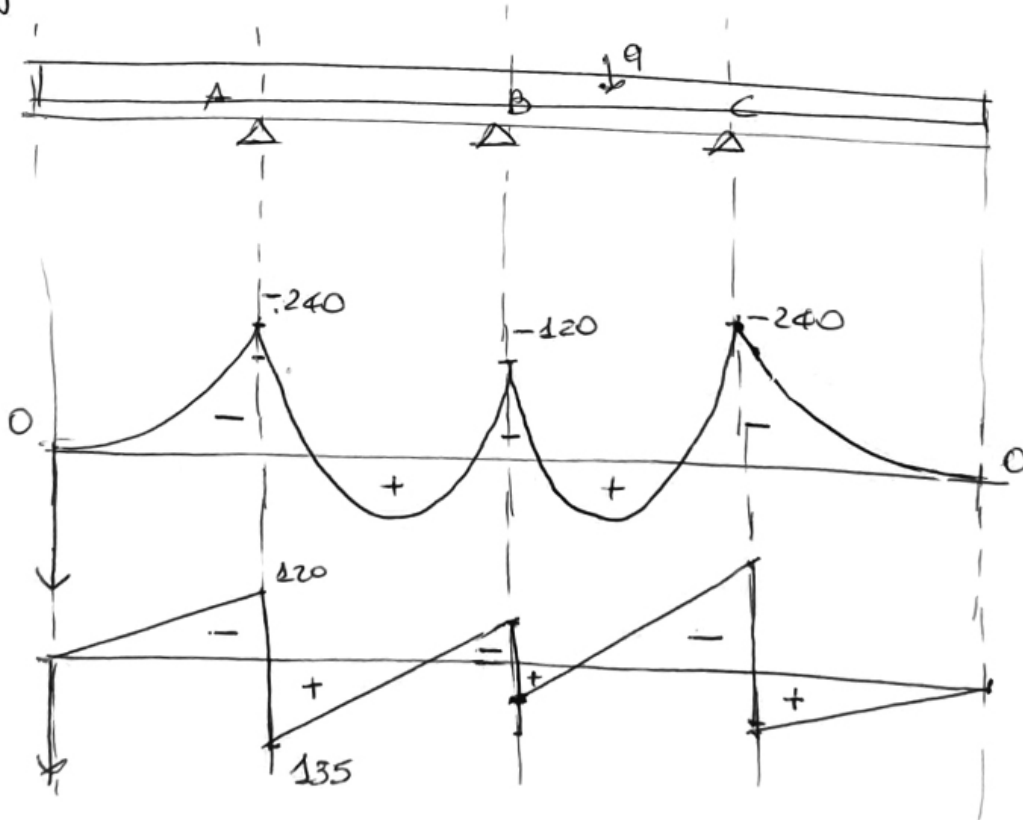
$$M_{FC} = 0$$

Table final moments

	A		B		C		
	E	AE	AB	BA	BC	CB	CF
μ	0	0	1	1/2	1/2	1	0
EOF	0			1/2	1/2		0
FEM	0	-240	+160	-160	+160	-160	+240
$-\Delta M$		+80	80			+80	
DIST							
EO	0			40	-40		
ΣM	0	-240	+240	-120	+120	-240	+240

5pt

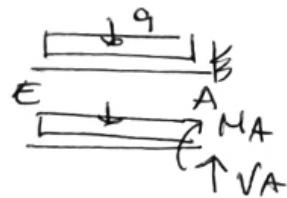
→ Diagrams



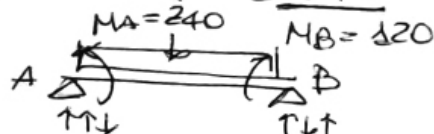
6pt

→ Shear force in A - Looking at left

$$V_{A-} = q l_1 = 30 \times 4 = 120 \text{ kN}$$



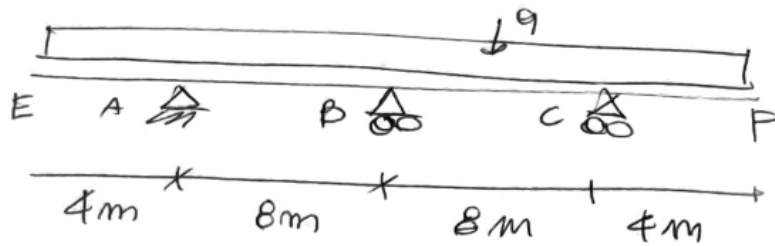
Looking at right:



$$V_{A+} = q \frac{l_2}{2} + \frac{M_A}{l_2} - \frac{M_B}{l_2} = \frac{30 \times 8}{2} + \frac{(240 - 120)}{8} = 135 \text{ kN}$$

5pt

EXERCISE 2 [35 pt] - OPTION 2

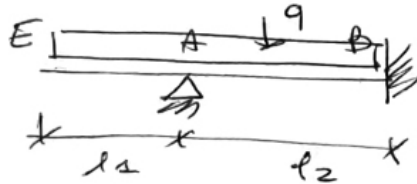


$$q = 30 \text{ kN/m}$$

$$l_1 = 4 \text{ m}$$

$$l_2 = 8 \text{ m}$$

Since the beam is symmetric, I can consider half:



→ Stiffness factors [in A]:

$$K_{AE} = 0$$

$$K_{AB} = \frac{4EI}{l_2} = \frac{4EI}{8} = \frac{1}{2} EI$$

→ Distribution factor: $\mu_{AE} = \frac{K_{AE}}{\sum K_A} = 0$

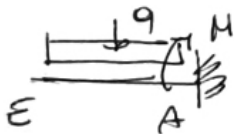
$$\mu_{AB} = \frac{K_{AB}}{\sum K_A} = \frac{\frac{1}{2} EI}{\frac{1}{2} EI} = 1$$

→ Carry-over factors:

$$M_{EA} = 0$$

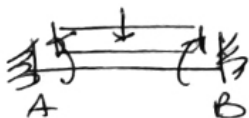
$$M_{BA} = \frac{1}{2} M_{AB}$$

→ Fixed-end moments:



$$M_{AE} = \frac{1}{2} q l_1^2 = -\frac{1}{2} \times 30 \times 4^2 = -240 \text{ kNm}$$

$$M_{EA} = 0$$



$$M_{AB} = \frac{1}{12} q l_2^2 = \frac{1}{12} \times 30 \times 8^2 = 160 \text{ kNm}$$

$$M_B = -160 \text{ kNm}$$

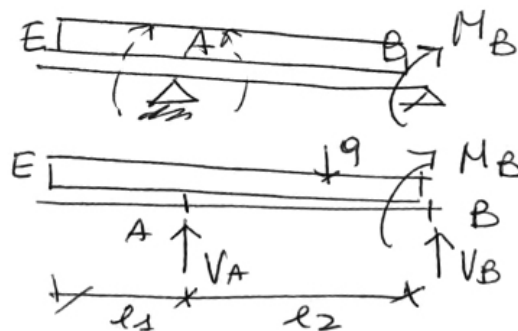
table final moments

		A		
	E	AE	AB	B
μ		0	1	
COF	0			1/2
FEM	0	-240	+160	-160
$-\Delta M$			+80	
DIST			80	
COF	0	0		+40
$\sum M$	0	-240	+240	-120

→ Diag rows

[Look at solution 1]

→ Shear force in A :

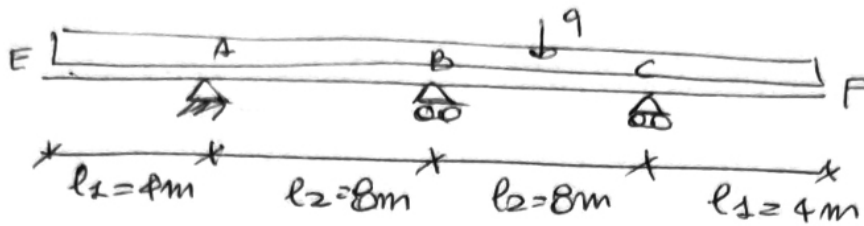


$$\sum M(B) = M_B + V_A l_2 - q \frac{(l_1 + l_2)^2}{2} = 0$$

$$V_A = \frac{-M_B + q \frac{(l_1 + l_2)^2}{2}}{l_2}$$

$$= \frac{-120 + 30 \frac{(4+8)^2}{2}}{8} = 255 \text{ kN}$$

EXERCISE 2 [35 pt] - OPTION B



$$q = 30 \text{ kN/m}$$

$$l_1 = 4 \text{ m}$$

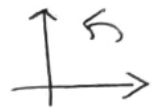
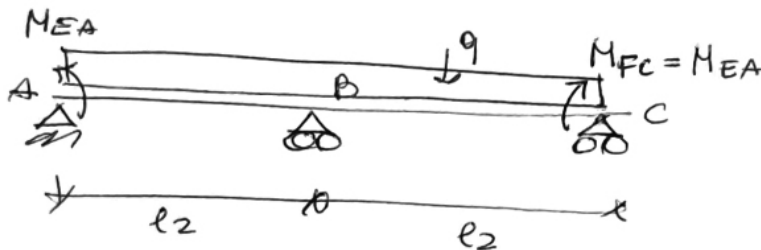
$$l_2 = 8 \text{ m}$$

EA and CF are cantilevers:

A moment M_{EA} is created because of this in A:

$$M_{EA} = \frac{1}{2} q l_1^2 = \frac{1}{2} \times 30 \times 4^2 = 240 \text{ kNm}$$

and similarly in C - So the beam can be simplified as:



Point of interest is B.

→ Stiffness factors : $k_{BA} = \frac{3EI}{l_2} = \frac{3}{8} EI$

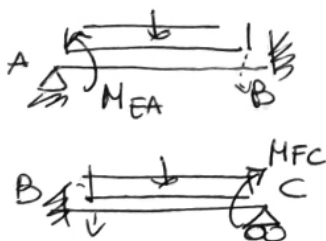
$$k_{BC} = \frac{3EI}{l_2} = \frac{3}{8} EI$$

→ Distribution factors :

$$\mu_{BA} = \frac{k_{BA}}{\sum k_B} = \frac{\frac{3}{8} EI}{(\frac{3}{8} + \frac{3}{8}) EI} = \frac{1}{2} = \mu_{BC}$$

→ Carry-over factors : $M_{AB} = 0 = M_{CB}$

→ Fixed-end moments



$$M_{BA}(q) = -\frac{q l^2}{8} = -\frac{30 \times 8^2}{8} = -240 \text{ kNm}$$

$$M_{BA}(M_{EA}) = \frac{1}{2} M_{EA} = +120 \text{ kNm}$$

$$M_{BC} = +\frac{q l^2}{2} = +240 \text{ kNm}$$

$$M_{BC}(M_{FC}) = \frac{1}{2} M_{FC} = -120 \text{ kNm}$$

$$M_{AB} = M_{EA} = +240 \text{ kNm}$$

$$M_{CB} = M_{FC} = -240 \text{ kNm}$$

$$M_{BA} = -120 \text{ kNm}$$

$$M_{BC} = +120 \text{ kNm}$$

Table for the final moments :

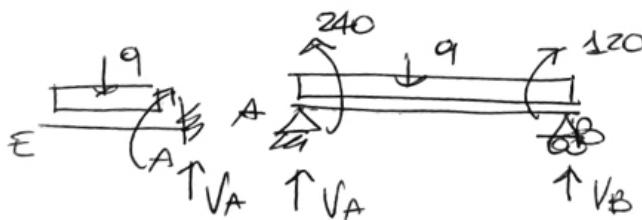
		B		
	A	BA	BC	C
or DOF		1/2	1/2	
	0			0
FEM	+240	+120	-120	-240
- ΔM			0	
ZM	+240	+120	-120	-240

→ Diagrams

[look at solution 1]

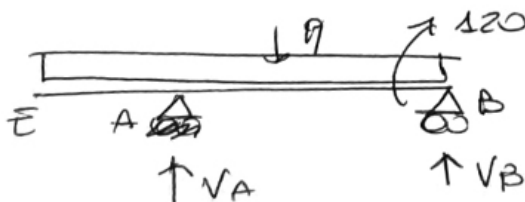
→ Shear force :

It can be obtained solving these beams:



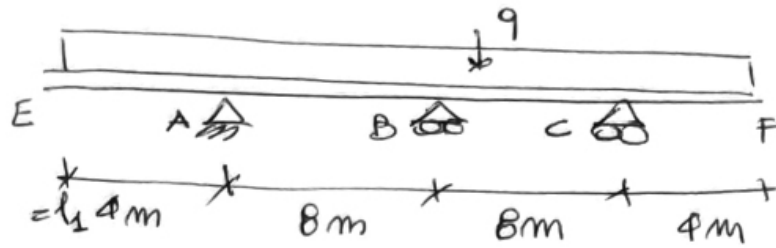
[like solution 1]

or



[like solution 2]

EXERCISE 2 → SOLUTION SIMPLIFIED OPTION 4

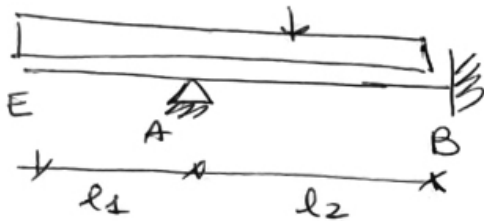


$$q = 30 \text{ kN/m}$$

$$l_1 = 4 \text{ m}$$

$$l_2 = 8 \text{ m}$$

The beam is symmetric, so I can consider $\frac{1}{2}$ beam with a fixed support in B (constraint for symmetry)



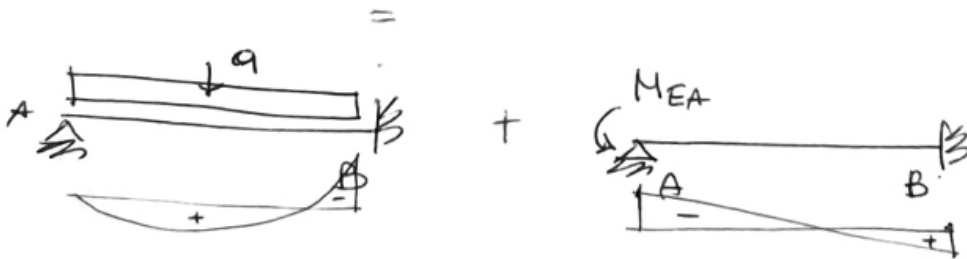
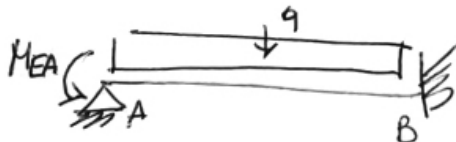
Moreover EA is a cantilever beam, so EA:



it means it creates a moment in A: $M_{EA} = \frac{1}{2} q l_1^2$

So the beam can be further simplified:

$$M_{EA} = \frac{1}{2} q l_1^2 = \frac{1}{2} 30 \times 4^2 = \underline{240 \text{ kNm}}$$



$$M_B = M_B(q) + M_B(M_{EA})$$

$$= -\frac{q l_2^2}{8} + M_{EA}/2 = -\frac{30 \times 8^2}{8} + 240/2 = -120 \text{ kNm}$$

$$M_A = 0 - M_{EA} = -240 \text{ kNm}$$

$$M_E \geq 0$$

For diagrams and shear force look at solution 2