

Course : Linear Algebra
Module : 2A
Course code : 202001205

Date : 9 May 2025
Time : 13:45 – 15:45
Reference : BIT, BMT, CE, CSE, IDE & TCS

Linear Algebra

Exam

Instructions

This exam contains 10 questions. You shall use the attached *answer form* to submit your answers.

- For questions 1–4, you are only required to fill in the **final answer** on the answer form.
- For questions 5–10, you are required to write down a **full calculation and argumentation**.

You will hand in your answer form only. If you run out of space, you can use the extra space at the end of the answer form. Refer clearly to that space in the original answer.

Do not write with red pen or pencil. Do not use correction fluid or tape.

The use of electronic devices is not allowed!

Final answer questions

Report all your answers on the answer form.

1. For each of the following statements, determine whether it is true or false. [3 pt]

- (a) For every square matrix A if the system $A\mathbf{x} = \mathbf{0}$ is consistent, then $\det(A) \neq 0$.
- (b) Any set containing the zero vector is linearly dependent.
- (c) Every subspace of $\mathbb{R}^2$ has exactly one basis.
- (d) For every matrix A if R is the reduced echelon form of A , then $\det(R) = \det(A)$.

2. Consider two lines in $\mathbb{R}^3$. The first line is given by $\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ and the second line [3 pt]

is given by $\left\{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \text{ for some } t \in \mathbb{R} \right\}$. Compute the intersection of the two lines.

3. Consider the matrix $A = \begin{pmatrix} 0 & 0 & c+1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$.

- (a) Find all $c \in \mathbb{R}$ such that A is invertible. [2 pt]
- (b) If $c = 5$, then compute A^{-1} . [2 pt]
- (c) For which $c \in \mathbb{R}$ are the columns of A linearly independent? [1 pt]

4. Let [2 pt]

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 4 \\ 0 & 6 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} -4 & 3 \\ 6 & -5 \end{pmatrix}.$$

Find a 2×2 matrix $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ such that

$$AX + XB = C.$$

Continues on the following page.

Open questions

The full solutions to questions 5–10 must be clearly written down on the answer form, including calculations and argumentations.

Points will not be awarded for reaching a correct result if this is not supported by a correct procedure and by a sound and clear argumentation.

5. Consider the matrices

[4 pt]

$$A = \begin{pmatrix} x & -1 \\ 2 & -2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & z \\ y & 1 \end{pmatrix}.$$

We are given that every element of $\text{Col } B$ belongs to $\text{Null } A$. Compute x, y, z .

6. Consider the subspace H of $\mathbb{R}^3$ given by $H = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} \right\}$.

(a) Find a basis for H .

[3 pt]

(b) Show that $H = \mathbb{R}^3$.

[1 pt]

7. Let A be a 3×3 matrix with eigenvalues 0, 3, 5 and corresponding eigenspaces

$$E_0 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}, \quad E_3 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}, \quad E_5 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(a) Briefly explain why $\text{Null } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

[1 pt]

(b) Find an invertible matrix Q and a diagonal matrix L such that $A^T = Q^{-1}LQ$.

[3 pt]

Hint: First find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

8. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation such that the area of the parallelogram spanned by the vectors $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ is equal to 2, where $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Denote by A the representation matrix of T .

(a) Is A invertible? Provide a brief explanation for your answer.

[1 pt]

(b) Assume that $A = \begin{pmatrix} 2 & 1 \\ 1 & x \end{pmatrix}$. Find all possible values of x .

[2 pt]

9. Let A be a matrix of size 3×4 and let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^4$.

(a) Show that if the vectors $\mathbf{v}$ and $2\mathbf{v}$ are both solutions to the system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b} = \mathbf{0}$.

[2 pt]

(b) Explain why the system $A\mathbf{x} = \mathbf{0}$ must have infinitely many solutions.

[2 pt]

10. Let $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be two linear transformations. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the function given by $T(\mathbf{x}) = T_1(\mathbf{x}) - T_2(T_1(\mathbf{x}))$.

(a) Prove that T is a linear transformation.

[3 pt]

(b) Prove that if $\mathbf{v} \in \ker(T_1)$, then $\mathbf{v} \in \ker(T)$.

[1 pt]

Total: 36 pt

1	2	3	4	5	6	7	8	9	10	Σ

Do not fill in this table.

A

Only for sorting.

Answer form

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Full name in BLOCK LETTERS _____

Student number _____

Programme _____

⚠ FILL IN YOUR DATA AS SOON AS YOU RECEIVE THE EXAM ⚠

Question 1. Check the correct answers.

(four correct answers= 3pt, three correct answers= 2pt, two correct answers= 1pt, up to one correct answer= 0pt)

- | | | |
|-----|-------------------------------|--------------------------------|
| (a) | ☐ True | ☐ False |
| (b) | ☐ True | ☐ False |
| (c) | ☐ True | ☐ False |
| (d) | ☐ True | ☐ False |

Question 2.

The intersection of the two lines is

Question 3.

(a) The matrix A is invertible when

(b) When $c = 5$ we have that A^{-1} is

(c) The columns of A are linearly independent when

Question 4.

$X =$

Question 5. Write a full calculation/argumentation clearly in the box.

[illegible]

[illegible][illegible]

[illegible][illegible]

[illegible][illegible]

[illegible][illegible]

[illegible][illegible]

[illegible]

[illegible]

[illegible]