

Grading scheme exam Vector Calculus Oct 7th 2022

1) The work is defined by $W = \int_C \mathbf{F} \cdot d\mathbf{s} = \textcircled{0.5}$ for defining work as a line integral $\textcircled{1}$

with C parametrized by $\mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} - t \mathbf{k}$
for $0 \leq t \leq 6\pi$.

Using that

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (-mg\mathbf{k}) \cdot (\cos t \mathbf{i} - \sin t \mathbf{j} - \mathbf{k}) = mg, \quad \textcircled{1} \text{ for correct integrand}$$

we obtain that

$$\textcircled{1} \quad \textcircled{0.5} = \int_0^{6\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{6\pi} mg dt = 6\pi mg. \quad \textcircled{0.5}$$

for writing a line integral
as a simple integral using
a parametrization

for correct
evaluation of
simple integral.

total 3 pts

2) a) A potential $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfies $\nabla f = \mathbf{F}$, i.e., the following three conditions must hold:

$$(i) \quad \frac{\partial f}{\partial x} = (1+x) e^{x+y}$$

$$(ii) \quad \frac{\partial f}{\partial y} = x e^{x+y} + 2y$$

$$(iii) \quad \frac{\partial f}{\partial z} = -2z$$

$\textcircled{0.5}$ for the definition of
potential (equivalently conditions
(i)-(iii))

From (i), we obtain that $f(x,y,z) = x e^{x+y} + g(y,z)$, $\textcircled{0.5}$ for correct x-int.

which together with (ii) implies that

$$xe^{x+y} + \frac{\partial g}{\partial y}(y,z) = xe^{x+y} + 2y, \text{ i.e. } \frac{\partial g}{\partial y}(y,z) = 2y.$$

Integration w.r.t. y yields $g(y,z) = y^2 + h(z)$, and

$$f(x,y,z) = xe^{x+y} + y^2 + h(z) \quad (0.5) \text{ for correct } y\text{-integration}$$

Condition (iii) then implies that $h'(z) = -2z$, i.e., $h(z) = -z^2 + c, c \in \mathbb{R}$.

(0.5) for correct z -integration

(0.5) for writing f .

The function $f(x,y,z) = xe^{x+y} + y^2 - z^2$ satisfies (i)-(iii),

i.e., f is a potential, because

(0.5) for verification (at least a remark)

$$\frac{\partial f}{\partial x} = (1+x)e^{x+y}, \quad \frac{\partial f}{\partial y} = xe^{x+y} + 2y, \quad \frac{\partial f}{\partial z} = -2z.$$

$$b) \quad H = g - f = 2(z-y)j + 2(z-y)k$$

On the curve we have that $z-y=t$, and $dy = y'(t)dt = dt$, $dz = z'(t)dt = 2dt$

$$\int_C H \cdot dr = 2 \int_0^1 t \, dt + 2 \int_0^1 t \, 2dt = 6 \int_0^1 t \, dt = 3. \quad (0.5)$$

(1) for writing the line integral as a simple integral

(0.5) for introducing F

(0.5) linearity of integral

for correct evaluation

$$c) \quad \int_C g \cdot dr = \int_C (g - f + f) \cdot dr = \int_C (g - f) \cdot dr + \int_C f \cdot dr$$

$$\stackrel{(b)}{=} 3 + \int_C f \cdot dr = 3 + f(r(1)) - f(r(0)) = 3 + 1 - 4 = -e$$

Fundamental Theorem

for line integrals

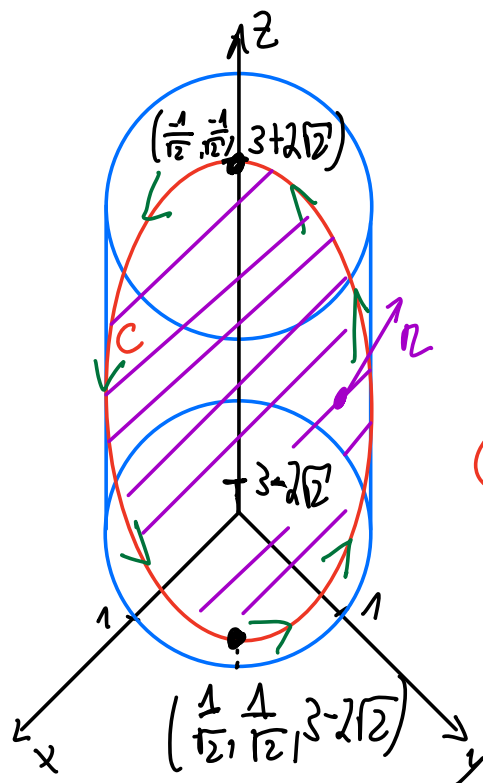
(1) for correct application

$$r(1) = j + 2k \\ r(0) = i$$

$$= -e. \quad (0.5) \text{ for correct evaluation}$$

$$\text{total: } 3 + 1.5 + 2.5 = 7$$

3)
a)



0.5 for cylinder and curve

0.25 for appropriate labelling

0.25 for orientation

/// shaded area optional
(see surface S in (b))

b) Circulation of $\vec{F} = \oint_C \vec{F} \cdot d\vec{r}$ 0.5 define circulation
0.5 stating Stokes

$$\text{Stokes' Thm: } \oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma = \textcircled{*}$$

with S being the portion of the plane $2x + 2y + z = 3$
enclosed by C .

Parametrization of C : $\vec{r}(r, \theta) = r \cos \theta \vec{i} + r \sin \theta \vec{j} + (3 - 2r(\cos \theta + \sin \theta)) \vec{k}$ 0.5

with $0 \leq \theta \leq 2\pi$ $0 \leq r \leq 1$.

for parameter domain 0.5

Then

$$\textcircled{*} = \pm \int_0^1 \int_0^{2\pi} (\nabla \times \vec{F}) \cdot (\vec{r}_r \times \vec{r}_\theta) \, d\theta \, dr = \textcircled{*}$$

1 for expressing surface integral as double integral, using a parametrization

Compute integrand:

$$\nabla \times F = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{pmatrix} = 0\mathbf{i} - 0\mathbf{j} + (3x^2 + 3y^2)\mathbf{k} \quad \text{computing curl} \quad (0.5)$$

$$\Rightarrow (\nabla \times F)(\vec{r}(H)) = 3r^2\mathbf{k} \quad (0.5) \text{ evaluation on curve}$$

$$\vec{r}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} - 2(\cos\theta + \sin\theta)\mathbf{k}$$

$$\vec{r}_\theta = -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j} + 2r(\cos\theta - \sin\theta)\mathbf{k}$$

(0.5) correct tangent vectors

$$\vec{r}_r \times \vec{r}_\theta = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2(\cos\theta + \sin\theta) \\ -r\sin\theta & r\cos\theta & -2r(\cos\theta - \sin\theta) \end{pmatrix}$$

$$= \begin{bmatrix} -2r\sin\theta(\cos\theta - \sin\theta) + 2r\cos\theta(\cos\theta + \sin\theta) \\ -[-2r\cos\theta(\cos\theta - \sin\theta) - 2r\sin\theta(\cos\theta + \sin\theta)] \\ + r\mathbf{k} \end{bmatrix} \mathbf{i} - \begin{bmatrix} -2r\cos\theta(\cos\theta - \sin\theta) - 2r\sin\theta(\cos\theta + \sin\theta) \end{bmatrix} \mathbf{j} + r\mathbf{k}$$

(0.5) correct cross product

$$= 2r\mathbf{i} + 2r\mathbf{j} + r\mathbf{k}$$

Because of the orientation of C (right-hand rule), the $\mathbf{k}$ -component of the normal must be positive.
Hence, we choose "+" in $(\star\star)$. (1) for determining $\pm$ in $(\star\star)$.

Thus

(0.5) computing inner product

$$(\star\star) = \int_0^1 \int_0^{2\pi} 3r^3 d\theta dr = 2\pi \left. \frac{3}{4}r^4 \right|_0^1 = \frac{3}{2}\pi \quad \text{correct value} \quad (0.5)$$

total: 1 + 7 = 8 pts