

Introduction To Mathematics & Calculus 1A

Exam

Instructions

- Motivate all your answers.
- The use of electronic devices or formula sheets is not allowed.
- Use the answer form to write down your answers. Clearly fill out your name, student number and study programme. Any text outside a frame will be ignored.
- For **Final Answer Questions**, on the answer sheet, in the corresponding text frame you provide only one answer. Do not write down a calculation, explanation or motivation. If you do write down a calculation, explanation or motivation, it will not be taken into account for grading.
- For **Open Questions**, on the answer sheet, you provide a full calculation or motivation in the text frame corresponding to the question. The calculation or motivation will be graded.
- If you need more space, you can write in the frame provided at the end of the answer form. Clearly refer to this space in the original answer.

Final answer questions Introduction to Mathematics

1. Let $A_k = [-k, -\frac{1}{k}]$ for $k \neq 0$.

(a) Determine

[1 pt]

$$\bigcup_{k=1}^{100} A_k.$$

(b) Determine

[1 pt]

$$\bigcap_{k=1}^{100} A_k.$$

2. Consider the statement $\neg(\forall x > 0(f(x) = 0 \rightarrow g(x) = 0))$ involving arbitrary functions $f(x), g(x)$. [2 pt]

Which of the following statements are logically equivalent to this statement?

There could be zero, one, or multiple equivalent statements.

All answers correct yields 2 points. 1 mistake yields 1 point

- (a) $\exists x > 0 (\neg(f(x) = 0 \rightarrow g(x) = 0))$
- (b) $\exists x \leq 0 (f(x) = 0 \rightarrow g(x) = 0)$
- (c) $\forall x > 0 (\neg(f(x) = 0 \rightarrow g(x) = 0))$
- (d) $\forall x \leq 0 (f(x) = 0 \rightarrow g(x) = 0)$
- (e) $\neg(\forall x > 0 (g(x) = 0 \rightarrow f(x) = 0))$
- (f) $\neg(\forall x > 0 (f(x) \neq 0 \rightarrow g(x) \neq 0))$

Open questions Introduction to Mathematics

3. For each of the following statements, either give a proof or a counterexample: [3 pt]

(a) For any set S of 3 different integers ($S \subseteq \mathbb{Z}, |S| = 3$), there exist $m, n \in S, m \neq n$ for which $m + n$ is even.

(b) For any set S of 3 different integers ($S \subseteq \mathbb{Z}, |S| = 3$), there exist $m, n \in S, m \neq n$ for which $m + n$ is odd.

4. For each integer $n \geq 0$, the *Fibonacci number* F_n is recursively defined by: [4 pt]

(1) $F_0 = 0, F_1 = 1,$

(2) for each $n \geq 2: F_n = F_{n-1} + F_{n-2}.$

Prove the following statement with mathematical induction:

$$\text{For all non-negative integers } n \geq 0: \sum_{i=0}^n F_i = F_{n+2} - 1.$$

5. Consider the set A of 4-digit numbers consisting of digits from $\{1, \dots, 8\}$, and containing each digit at most once. Provide a full solution, including computations and motivation.

(a) How many numbers in A consist of exactly 2 even digits and 2 odd digits? [1.5 pt]

(b) How many numbers in A satisfy **at least** 1 of the following conditions: [1.5 pt]

- It consists of exactly 2 even digits and 2 odd digits.

Final answer questions Calculus 1A

6. Define $P(1, 0, -1)$ and the vectors $\mathbf{u} = (3, -1, 2)$ and $\mathbf{v} = (-1, 1, -1)$.

- (a) Calculate $\mathbf{u} \times \mathbf{v}$. [2 pt]
- (b) Find an equation for the plane M that contains the point P and that is parallel to $\mathbf{u}$ and $\mathbf{v}$. Give the equation in the following form: $ax + by + cz = d$. [2 pt]
- (c) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$. [2 pt]

Open questions Calculus 1A

7. For which values of a and b is the function

[3 pt]

$$f(x) = \begin{cases} ax + b & \text{if } x \leq 0, \\ \frac{x}{\sqrt{x+1}-1} & \text{if } x > 0. \end{cases}$$

continuous for every x ?

8. Calculate the following limit :

[3 pt]

$$\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{1 - \cos(x)}$$

9. Define the function

$$g(x) = \sqrt[3]{x^2 - 1}.$$

- (a) Find all critical points of g . [2 pt]
- (b) Find the absolute extreme values of g on the interval $[-3, 3]$. [2 pt]

10. Define

$$h(x, y) = \frac{x^2 - y}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$.

- (a) Calculate [3 pt]

$$\lim_{(x, y) \rightarrow (0, 0)} h(x, y),$$

or show that this limit does not exist.

- (b) Find an equation for the plane tangent to the graph of $h(x, y)$ at the point $(1, 1, 0)$. [3 pt]