

Course : **Mathematics B2 (Newton)**

Date : January 8, 2016

Time : 13.45 - 15.45

**Motivate all answers and calculations.
The use of electronic devices is not permitted.**

[3p] 1. a) Show through explicit calculation that

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1} - 1} = 2$$

Idea to use l'Hopital's rule and check '0/0'

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1} - 1} \\ &= \lim_{x \rightarrow 0} \frac{1}{\frac{1}{2}(x+1)^{-1/2}} \\ &= 2 \end{aligned}$$

Alternatively, using the 'root formula'

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1} - 1} \\ &= \lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1} - 1} \cdot \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} \\ &= \lim_{x \rightarrow 0} \frac{x(\sqrt{x+1} + 1)}{x} \\ &= \lim_{x \rightarrow 0} (\sqrt{x+1} + 1) = 2 \end{aligned}$$

[2p] b) For which real values of p and q is the function

$$f(x) = \begin{cases} px + q & \text{als } x \leq 0 \\ \frac{x}{\sqrt{x+1} - 1} & \text{als } x > 0 \end{cases}$$

continuous at every point x ?

Continuity at a point $x = a$ requires

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

This is readily verified for all $a \neq 0$.

Continuity in $x = 0$ likewise requires

$$f(0) = \lim_{x \rightarrow 0^-} f(x) = q = \lim_{x \rightarrow 0^+} f(x) = 2$$

Hence, continuity of f is achieved at all $x \in \mathbb{R}$ for $q = 2$

and arbitrary $p \in \mathbb{R}$

- 2.** The function f is given by $f(x) = \tan^{-1}(x^2)$.

[2p] a) Determine $f'(x)$.

The derivative of $g(t) = \tan^{-1}(t)$ is $g'(t) = 1/(1 + t^2)$.

Using this and the chain rule we find

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\tan^{-1}(x^2) \right) = \frac{1}{1 + (x^2)^2} \cdot (2x) \\ &= \frac{2x}{1 + x^4} \end{aligned}$$

[2p] b) Determine the linearisation of $f(x)$ in $x = 1$.

Noting that $f(1) = \tan^{-1}(1) = \pi/4$ and $f'(1) = 2/(1 + 1) = 1$

we may write for the linearisation $L(x)$ of $f(x)$ at $x = 1$:

$$L(x) = f(1) + f'(1)(x - 1) = \frac{\pi}{4} + (x - 1)$$

- [4p] **3.** Determine all extreme values (absolute and local) of the function $f(x) = x \ln(x)$ on the interval $(0, 5]$.
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We observe

$$f'(x) = \ln(x) + 1 = 0$$

in $x^* = e^{-1} \in (0, 5]$

Since $f'(x) < 0$ for $0 < x < x^*$ and $f'(x) > 0$ for $x > x^*$ we note that f has a global minimum in x^*

equal to $f(x^*) = -e^{-1} < 0$.

In addition, f has a local maximum at $x = 5$ given by $f(5) = 5 \ln(5) > 0$.

Finally, the limit value as $x \rightarrow 0^+$ is found as (use l'Hopital)

$$\lim_{x \rightarrow 0^+} x \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} -x = 0$$

This in turn implies that f has a global maximum at $x = 5$.

4. Given

$$f(x, y) = \frac{x^2 - y}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$.

[2p] a) Compute

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$$

or show that this limit does not exist.

The limit does not exist - as an example we note a 'conflicting' situation

$$\lim_{x \rightarrow 0} f(x, 0) = \lim_{x \rightarrow 0} \frac{x^2}{x^2} = 1 \neq \lim_{y \rightarrow 0^-} f(0, y) = \lim_{y \rightarrow 0^-} \frac{-y}{y^2} \rightarrow \infty$$

Since the result depends on the path taken to $(0,0)$ the corresponding limit does not exist.

Alternatively, using polar coordinates: $x = r \cos(\theta)$ and $y = r \sin(\theta)$ we observe

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{r \rightarrow 0^+} \frac{r^2 \cos^2(\theta) - r \sin(\theta)}{r^2} = \lim_{r \rightarrow 0^+} \left(\cos^2(\theta) - \frac{1}{r} \sin(\theta) \right)$$

The result depends on θ and does not yield a unique value.

[3p] b) Determine the equation of the plane tangent to the graph of $f(x, y)$ at the point $(1, 1, 0)$.

We may compute

$$\frac{\partial}{\partial x} (f(x, y)) = \frac{2x(x^2 + y^2) - (x^2 - y)(2x)}{(x^2 + y^2)^2} = \frac{2xy(y + 1)}{(x^2 + y^2)^2} \quad ; \quad \frac{\partial}{\partial x} (f(1, 1)) = \frac{4}{4} = 1$$

and

$$\frac{\partial}{\partial y} (f(x, y)) = \frac{-(x^2 + y^2) - (x^2 - y)(2y)}{(x^2 + y^2)^2} \quad ; \quad \frac{\partial}{\partial y} (f(1, 1)) = -\frac{2}{4} = -\frac{1}{2}$$

Hence, the equation for the tangent plane at $(1, 1, 0)$ is

$$z = f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) \quad \rightarrow \quad z = (x - 1) - \frac{1}{2}(y - 1)$$

since $f(1, 1) = 0$.

5. Determine $\frac{dy}{dx}$ in case $f(t)$ is a continuous function on $\mathbb{R}$ and

[2p]

a)

$$y(x) = \int_{e^x}^0 f(t) dt$$

Applying the computational rules we infer

$$\frac{dy}{dx} = -f(e^x) \cdot e^x$$

Sign correct ([1/2 p]), factor $f(e^x)$ [1p], factor e^x .

[2p]

b)

$$y(x) = \int_x^0 f(e^t) dt$$

Applying the computational rules we infer

$$\frac{dy}{dx} = -f(e^x)$$

[3p]

6. a) Compute

$$\int_0^{\frac{1}{2} \ln(3)} \frac{e^{2x}}{1 + e^{2x}} dx$$

We substitute $u = e^{2x}$,

for which $du = 2e^{2x} dx$

and $u(0) = 1$, $u(\frac{1}{2} \ln(3)) = 3$

$$\begin{aligned} \int_0^{\frac{1}{2} \ln(3)} \frac{e^{2x}}{1 + e^{2x}} dx &= \int_1^3 \frac{1/2}{1 + u} du = \frac{1}{2} \ln(1 + u) \Big|_1^3 \\ &= \frac{1}{2} (\ln(4) - \ln(2)) = \frac{1}{2} (2 \ln(2) - \ln(2)) = \frac{1}{2} \ln(2) \end{aligned}$$

[3p]

b) Determine

$$\int x \sin(2x + 1) dx$$

Integration by parts yields

$$\begin{aligned} \int x \sin(2x + 1) dx &= -\frac{1}{2} x \cos(2x + 1) - \int 1 \cdot \left(-\frac{1}{2} \cos(2x + 1)\right) dx \\ &= -\frac{1}{2} x \cos(2x + 1) + \frac{1}{4} \sin(2x + 1) + C \end{aligned}$$

with $C \in \mathbb{R}$ an arbitrary constant.

[3p] c) Compute

$$\int_0^4 \frac{1}{\sqrt{4-x}} dx$$

Improper integral

$$\begin{aligned} \int_0^4 \frac{1}{\sqrt{4-x}} dx &= \lim_{t \rightarrow 4} \int_0^t \frac{1}{\sqrt{4-x}} dx \\ &= \lim_{t \rightarrow 4} (-2\sqrt{4-x}) \Big|_0^t \\ &= \lim_{t \rightarrow 4} -2(\sqrt{4-t} - 2) = 4 \end{aligned}$$

[2p] 7. a) Determine the $x \in \mathbb{R}$ for which the series

$$\sum_{n=0}^{\infty} (x^2 - 2)^n$$

is convergent. For those x -s determine the corresponding infinite sum.

This is a geometric series in the variable $y = x^2 - 2$, resulting in an infinite sum $1/(1 - y) = 1/(1 - x^2 + 2) = 1/(3 - x^2)$, provided the series converges.

It is convergent iff $|y| < 1$, i.e., $|x^2 - 2| < 1$, implying $-1 < x^2 - 2 < 1$, i.e., $1 < x < \sqrt{3}$ as well as $-\sqrt{3} < x < -1$.

[3p] b) Compute the Taylor polynomial of order 2 in $x = 0$ of the function $f(x) = \ln(1 - x^2)$.

The desired Taylor polynomial of order 2, $p_2(x)$ can be expressed as

$$p_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$$

Since $f(0) = \ln(1) = 0$ and

$$f'(x) = \frac{-2x}{1-x^2} \quad ; \quad f'(0) = 0$$

and

$$f''(x) = \frac{-2(1-x^2) - (-2x)(-2x)}{(1-x^2)^2} = \frac{-2-2x^2}{(1-x^2)^2} \quad ; \quad f''(0) = -2$$

we obtain $p_2(x) = -x^2$.

Total: 36 points