

Course : Calculus 2  
 Module : 2A  
 Course code : 202001218

Date : Friday 21 April, 2023  
 Time : 13:45 – 15:45 (2 hours)  
 Reference : ME-CAL2-2223-resit

## Calculus 2

### Resit

### Instructions

This exam contains 10 exercises. You shall use the provided *answer form* to submit your answers.

- For exercises 1–6, you are only required to fill in the **final answer** on the answer form.
- For exercises 7–10, you are required to write down a **full calculation and argumentation**.

You will hand in your answer form only. **Any text outside the answer form will not be considered.**

If you run out of space, you can use the extra space at the end of the answer form. Refer clearly to that space in the original answer.

\*\*\* The use of electronic devices is not allowed. \*\*\*

### Final answer questions

1. Let  $\mathbf{r}(t) = \cos\left(\frac{\pi}{4}t\right)\mathbf{i} + \sin\left(\frac{\pi}{4}t\right)\mathbf{j} + \left(t^2 - \frac{1}{4}\right)\mathbf{k}$  represent the position vector of a particle at time  $t$ . [2 pt]

Which of the three statements below is/are **True**?

Report your answer on the answer form.

- (1)  $\mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt} = 0$  for all  $t$ .
- (2)  $\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}$  is equal to the the square of the speed.
- (3) The tangent line at  $\mathbf{r}(1)$  is  $\{\mathbf{r}(1) + s \cdot \mathbf{r}'(1) : s \in \mathbb{R}\}$

- |                                                                  |                                                             |
|------------------------------------------------------------------|-------------------------------------------------------------|
| <input type="checkbox"/> A (1), (2), and (3) all are <b>True</b> | <input type="checkbox"/> E (1) and (2) only are <b>True</b> |
| <input type="checkbox"/> B none of them is <b>True</b>           | <input type="checkbox"/> F (1) only is <b>True</b>          |
| <input type="checkbox"/> C (2) and (3) only are <b>True</b>      | <input type="checkbox"/> G (2) only is <b>True</b>          |
| <input type="checkbox"/> D (1) and (3) only are <b>True</b>      | <input type="checkbox"/> H (3) only is <b>True</b>          |

2. Let  $f(x, y) = \ln(\cos(x) + \sin(y) + 1)$ .

(a) Let  $L(x, y)$  denote the linearization of  $f(x, y)$  at  $(\frac{\pi}{2}, 0)$ . Determine  $L(x, y)$ . [2 pt]

(b) Let  $P_2(x, y)$  denote the second-degree Taylor polynomial of  $f(x, y)$  at  $(\frac{\pi}{2}, 0)$ . Determine the coefficient of the  $x^2$  term in  $P_2(x, y)$ . [1 pt]

Report your answer on the answer form.

3. Let  $f(x, y)$  be a function such that

[3 pt]

$$\nabla f = (3x^2 - ky) \mathbf{i} + (3y^2 - kx) \mathbf{j},$$

where  $k$  is a real number.

Which of the following three statements about the given critical points of  $f$  is/are **True**?

Report your answer on the answer form.

- (1) If  $k \neq 0$ , then  $(0, 0)$  is a saddle point.
- (2) If  $k > 0$ , then  $\left(\frac{k}{3}, \frac{k}{3}\right)$  is a local maximum.
- (3) If  $k > 0$ , then the critical point  $\left(\frac{k}{3}, \frac{k}{3}\right)$  is an interior point in the region in the first quadrant bounded by the  $x$ -axis, the line  $y = x$  and the line  $x = k$ .

4. The region  $E$  of the Cartesian space is bounded below by the plane  $z = 0$ , above by the plane  $z = 1$ , and sideways by the sphere of radius 2 centered at the origin.

[3 pt]

Exactly **two** of the expressions A—H below correctly represent the volume of  $E$ : one is in spherical coordinates, and the other one is in cylindrical coordinates.

Find them and report your answer on the answer form.

☐ A  $\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-\rho^2}} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$

☐ B  $4 \int_0^{\pi/2} \int_0^{\sqrt{3}} \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$

☐ C  $4 \int_0^{\pi/2} \int_{\pi/3}^{\pi/2} \int_0^2 \rho^2 \cos \varphi \, d\rho \, d\varphi \, d\theta$

☐ D  $6 \int_0^{\pi/3} \int_{\pi/3}^{\pi/2} \int_0^1 \rho^2 \cos \varphi \, d\rho \, d\varphi \, d\theta$

☐ E  $2 \int_0^\pi \int_0^1 \int_0^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta$

☐ F  $\int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta + \int_0^{2\pi} \int_{\sqrt{3}}^2 \int_0^{\sqrt{r^2-4}} r \, dz \, dr \, d\theta$

☐ G  $\int_0^{2\pi} \int_0^{\pi/3} \int_0^{1/\cos \varphi} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta + \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$

☐ H  $2 \left( \int_0^\pi \int_0^1 \int_0^{\sqrt{3}} r \, dr \, dz \, d\theta + \int_0^\pi \int_1^2 \int_0^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta \right)$

5. The expression below represents the double integral of the function  $f(x, y) = e^{x+2y}$  over a region  $R$  in the Cartesian plane. Sketch the region  $R$  of integration.

[2 pt]

$$\int_0^2 \int_{x-2}^{2-x} e^{x+2y} \, dy \, dx + \int_1^3 \int_{-x}^{x-2} e^{x+2y} \, dy \, dx$$

Use the space provided on the answer form. Do not evaluate the expression.

6. The region  $R$  in the first quadrant of the Cartesian plane is bounded above by the line  $y = x$  [3 pt]  
and below by the circle of equation  $x^2 + (y - 1)^2 = 1$ .

Express  $\iint_R 2xy \, dA$  as an iterated double integral in polar coordinates.

Report your answer on the answer form. Do not evaluate the integral.

## Open questions

The full solutions to exercises 7—10 must be clearly written down on the answer form (including calculations and argumentation).

7. The Celsius temperature in a region of space is given by the formula

$$T(x, y, z) = e^{-x^2 + 3y^2 - 2z^2}.$$

- (a) Determine the direction of fastest temperature increase at the point  $(1, 1, 1)$ . [2 pt]

- (b) Assume that a curve of points  $(x, y, z)$  is parametrized by the time  $t$  as follows: [2 pt]

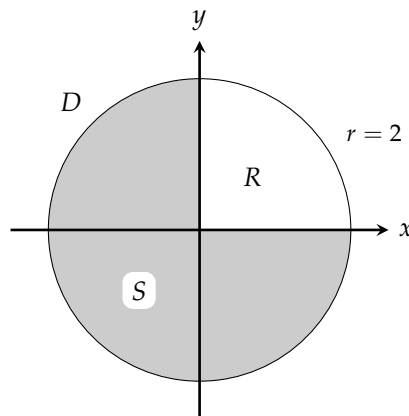
$$(x, y, z) = (t^2, \cos(t - 1), t^3).$$

Evaluate the rate of change in the temperature with respect to the time  $t$  at  $t = 1$ .

- (c) Consider the region of space  $S$  that has fixed temperature  $25^\circ$  Celsius. What is the [1 pt]  
relationship between  $S$  and the gradient vector  $\nabla T$  at a point  $(x_0, y_0, z_0)$  on  $S$ ?

8. Use the method of Lagrange multipliers to find the extreme values of the function [5 pt]  
 $f(x, y, z) = 2x + 8y + 6z$  subject to the constraint  $x^2 + y^2 + z^2 = 26$ .

9. The figure below shows a thin circular lamina  $D$  of radius 2 centered at the origin of a Cartesian plane. The lamina is composed of two parts  $R$  and  $S$  of different materials: the region [5 pt]  
 $R$  (white in the figure) is the part of the lamina on the first quadrant, while the remaining region  $S$  (shaded) is the part lying on the other three quadrants.



The density of the lamina  $D$  is given by the following density function:

$$\delta(x, y) = \begin{cases} 1 & \text{on the region } R, \\ 1 + \sqrt{x^2 + y^2} & \text{on the region } S. \end{cases}$$

Furthermore, it is known that the total mass of  $D$  is  $8\pi$ .

Find the coordinates  $(\bar{x}, \bar{y})$  of the center of mass of  $D$ .

- 10.** The triangle  $T$  on the Cartesian plane is bounded by the  $x$ -axis, the  $y$ -axis, and the line  $y + x = 1$ .

Consider the transformation given by  $u = y - x$  and  $v = y + x$ .

- (a) Find the absolute value of the Jacobian  $J(u, v)$ . [2 pt]
- (b) Find the image of  $T$  under the transformation given above. [1 pt]  
Draw the transformed region on the  $(u, v)$ -plane.
- (c) Compute the double integral: [2 pt]

$$\iint_T 4 \cdot e^{(y-x)/(y+x)} dA$$

|   |   |   |   |   |   |   |   |   |    |
|---|---|---|---|---|---|---|---|---|----|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|   |   |   |   |   |   |   |   |   |    |

Do not write above this line Do not write above this line Do not write above this line Do not write above this line Do not write above this line Do not write above this line Do not write above this line

# Answer Form

Calculus 2 – Resit – Friday 21 April, 2023

Full name in BLOCK LETTERS \_\_\_\_\_

Student number \_\_\_\_\_

▲ ▲ ▲ DO NOT FORGET TO FILL IN YOUR DATA ▲ ▲ ▲

**Question 1.** Mark your answer with a cross.

☐ A ☐ B ☐ C ☐ D ☐ E ☐ F ☐ G ☐ H

**Question 2(a).**  $L(x, y) =$

**Question 2(b).** The coefficient of the  $x^2$  term in  $P_2(x, y)$  is

**Question 3.** Mark your answer with a cross.

Statement **(1)** is: ☐ True ☐ False

Statement **(2)** is: ☐ True ☐ False

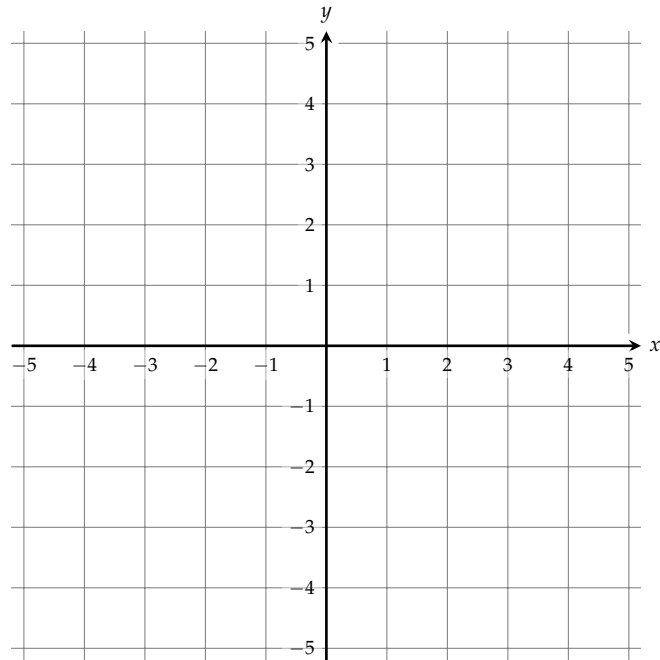
Statement **(3)** is: ☐ True ☐ False

**Question 4.** Mark with a cross the two expressions for the volume of  $E$ .

*Note: if more than two options are chosen, the answer is invalid (0 pt).*

☐ A   ☐ B   ☐ C   ☐ D   ☐ E   ☐ F   ☐ G   ☐ H

**Question 5.** Draw the region of integration.



**Question 6.**  $\iint_R 2xy \, dA$  as an iterated double integral in polar coordinates is

[illegible][illegible][illegible]

[illegible]

[illegible]

**Question 10(a).** Write a full calculation/argumentation clearly in the box.

|  |
|--|
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |

**Question 10(b).** Write a full calculation/argumentation clearly in the box.

|  |
|--|
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |

**Question 10(c).** Write a full calculation/argumentation clearly in the box.

|  |
|--|
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |
|  |

[illegible]

[illegible]