

The exam consists of 7 problems. Maximum scores are listed in the table below. The grade is determined as follows: $(\text{points} + 10) / 10$.

The use of a calculator is allowed, you are not allowed to use hand-written notes, smartphone, books, lecture slides, etc.

Good luck! Marc Uetz and Oskar Eikenbroek

problem	max score
1	8
2	12
3	16
4	8
5	19
6	16
7	11
total	90

1. [8 pts] Two questions about convexity of sets and functions.

- (a) [4 pts] Consider the sets $\mathcal{F}_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^3 + x_2^2 \geq 0\}$ and $\mathcal{F}_2 = \{(x_1, x_2) \in \mathbb{R}_{\geq 0}^2 \mid x_1^3 + x_2^2 \geq 0\}$. Prove or disprove if $\mathcal{F}_1$ is a convex set and if $\mathcal{F}_2$ is a convex set.

For disproving convexity, give two points $\mathbf{x}, \mathbf{y} \in \mathcal{F}$, and a convex combination of $\mathbf{x}$ and $\mathbf{y}$ that is not contained in $\mathcal{F}$.

- (b) [4 pts] Consider the function $z : \mathbb{R}^2 \rightarrow \mathbb{R}$, $z(\mathbf{x}) = e^{x_1} \cdot e^{x_2}$. Verify that z is globally convex. Is it also strictly convex? Prove or disprove.

2. [12 pts] Briefly (!) argue which of the following is either true or false, by either giving an argument or a counterexample. You may also use a sketch to support your argument(s).

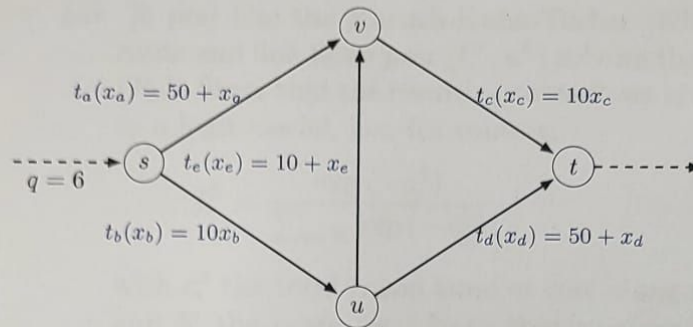
- (a) For any constrained minimization problem with convex objective function $z(\cdot)$ and convex feasible region, the step length in the direction of a feasible decent direction $\mathbf{d}$ at a feasible point $\mathbf{x}$ is *not* constrained by any of the active constraints at $\mathbf{x}$.
- (b) For any linearly constrained minimization problem with convex objective function $z(\cdot)$, when $\mathbf{x}^*$ is a locally optimal point, i.e., $z(\mathbf{x}^*) \leq z(\mathbf{x})$ for all feasible $\mathbf{x}$ with $\|\mathbf{x}^* - \mathbf{x}\| < \epsilon$ for some $\epsilon > 0$, then $\mathbf{x}^*$ must fulfill the KKT conditions.
- (c) For any linearly constrained minimization problem with convex objective function $z(\cdot)$, consider some feasible point $\mathbf{x}$ so that $\mathbf{x} - \epsilon \nabla z(\mathbf{x})$ is infeasible for all $\epsilon > 0$, then $\mathbf{x}$ is a locally optimal, and hence also a globally optimal point.
- (d) In a Wardrop model with linear link delay functions, assume we have computed a system optimum flow $\mathbf{f}$ that minimizes the total travel time. Then by linearity, flow $\mathbf{f}$ must also be a Wardrop equilibrium.

3. [16 pts] Consider the following linearly constrained, quadratic opti-

mization problem

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^2} \quad & z(\mathbf{x}) = (x_1 - 4)^2 + \left(x_2 + \frac{1}{2}\right)^2 - x_1 x_2 \\ \text{s.t.} \quad & g_1(\mathbf{x}) = x_1 \geq 0 \\ & g_2(\mathbf{x}) = x_2 \geq 0 \\ & g_3(\mathbf{x}) = -x_1 - 3x_2 + 9 \geq 0 \\ & g_4(\mathbf{x}) = -x_1 + 6 \geq 0 \end{aligned}$$

- (a) [6 pts] Make a sketch of the set of feasible solutions. Compute and depict the *unconstrained* optimum solution $\mathbf{x}^{**}$ (Sanity check: $\mathbf{x}^{**} = (5, 2)$ is outside the feasible region). From your sketch, make a rough guess where the constrained optimal solution $\mathbf{x}^*$ might be.
 - (b) [6 pts] Let us assume that someone has guessed that the constraints g_3 and g_4 are both tight at the optimum solution. Using the Karush-Kuhn-Tucker (KKT) conditions, verify if this guess is correct or not.
 - (c) [6 pts] Starting at $\mathbf{x}^0 = (0, 0)$, write down the linear programming problem (LP) for the first iteration of the Frank-Wolfe method for this problem. Then read off the optimal LP solution $\mathbf{y}$ from your sketch. Compute the resulting direction $\mathbf{d}$, the optimal step size α , and the next iterate $\mathbf{x}^1 = \mathbf{x}^0 + \alpha \mathbf{d}$.
4. [8 pts]
- We reconsider the same constrained quadratic optimization problem as in Problem 3.
- (a) [3 pts] Starting at point $\mathbf{x} = (4, 0)$, compute the steepest descent direction, as well as the direction that would result from using Frank-Wolfe. Compare and discuss.
 - (b) [5 pts] Compute the optimal solution $\mathbf{x}^*$ for the optimization problem of 3. (Note, $\mathbf{x}^*$ is not integer valued.)
5. [19 pts] Consider the network with corresponding link performance or latency functions in the figure below. In this network, a demand of $q = 6$ units of flow is to be sent from s to t .



Fisk proposed an extension of Beckmann's formulation, with a logit-based stochastic user equilibrium as optimal solution. Hence, where the original Beckmann problem can be formulated as

$$P_W : \quad \min_{(\mathbf{f}, \mathbf{x})} z(\mathbf{x}) \quad \text{s.t.} \quad (\mathbf{f}, \mathbf{x}) \in \mathcal{F}.$$

Fisk introduced mathematical optimization problem

$$P_S : \quad \min_{(\mathbf{f}, \mathbf{x})} z(\mathbf{x}) + \sum_{r \in \mathcal{R}} f_r (\ln(f_r) - 1) \quad \text{s.t.} \quad (\mathbf{f}, \mathbf{x}) \in \mathcal{F},$$

with $z(\mathbf{x})$ the Beckmann objective function, and f_r the route flow on route r (the route set is denoted by $\mathcal{R}$), and $\mathcal{F}$ the feasible set (of route and link flows). For completeness, we need to define $0 \ln(0) = 0$.

- (a) [3 pts] Assuming initial link flows $\mathbf{x}^0 = \mathbf{0}$ (i.e., zero for each link), provide the route and link flows assuming an all-or-nothing traffic assignment. Calculate the resulting travel times after AON assignment for all the routes in the network.
- (b) [4 pts] Provide Beckmann's and Fisk's optimization problem (P_W) and (P_S), respectively, tailored to the specific network presented in the figure above.
- (c) [6 pts] Use the Karush-Kuhn-Tucker (KKT) conditions to find route and link flows pair $(\mathbf{f}^W, \mathbf{x}^W)$ solving the optimization problem (P_W)

- (d) [6 pts] Use the Karush-Kuhn-Tucker (KKT) conditions to find route and link flows pair $(\mathbf{f}^S, \mathbf{x}^S)$ solving the optimization problem (P_S) . Show that the resulting route flows are distributed according to a logit model, i.e., for route r ,

$$f_r^S = \frac{\exp(-c_r^S)}{\sum_{p \in \mathcal{R}} \exp(-c_p^S)},$$

with c_r^S the total travel time or cost along route r in equilibrium, and $\mathcal{R}$ the route set. Note that you can assume here that the f -part of the optimal solution $(\mathbf{f}^S, \mathbf{x}^S)$ is so that $f_r^S > 0$.

Hint: use the fact that $\exp(\ln(x)) = x$, for $x > 0$.

6. [16 pts]

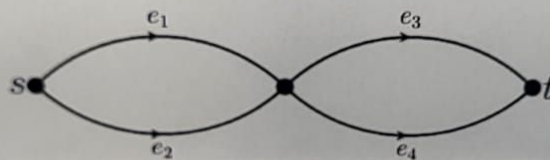
Streets and roadways cannot carry more than a certain amount of vehicular flow known as "capacity". In a traffic assignment context, each network link $a \in A$ (A is the set of links) is then associated with a capacity $c_a \geq 0$, and feasible link flows x_a should - in addition to the conditions of an uncapacitated traffic assignment - satisfy $x_a \leq c_a$.

As an alternative to Wardrop's condition, Correa et al. (2004) defined a condition, stating that *A feasible flow $\mathbf{f}$ represents a (capacitated) user equilibrium if no OD pair has an unsaturated path with strictly smaller cost than any path used for that pair.*

Here, a path p is said to be unsaturated with respect to a given feasible flow if and only if $x_a < c_a$ for all arcs a in path p , i.e., if the link flows are strictly less than capacity for all links of that path.

- (a) [4 pts] Provide (visually) an example of a traffic network (including a graph representation, an origin-destination matrix, and link performance functions) with a flow in Wardrop (user) equilibrium, and illustrate - using the definition above - that this distribution of traffic can be considered to be in equilibrium following the condition of Correa et al.
- (b) [4 pts] Provide (visually) an example of a traffic network (including a graph representation, OD matrix, link performance functions, link capacities) with a feasible flow (satisfying the capacity constraint) that satisfies the condition of Correa equilibrium but does not satisfy the Wardrop condition.

- (c) [8 pts] Using the ideas outlined in the description of the feasible direction methods a modification to the convex combinations method was developed so that the algorithm would solve the capacity constrained UE problem (i.e., Beckmann's problem with the capacity constraint introduced above). Devise a method for finding an initial feasible solution for such an algorithm.
7. [11 pts] Under natural assumptions regarding link performance functions, the edge or link flow part $\mathbf{x}^*$ of a Wardrop equilibrium $(\mathbf{f}^*, \mathbf{x}^*)$ is uniquely determined. This does not hold for the path flow part $\mathbf{f}^*$. Consider the example network in the Figure below, with 4 edges e_j , $j = 1, \dots, 4$.



Let the strictly increasing edge costs be

$$t_{e_j}(x_{e_j}) = 1 + x_{e_j}, \quad j = 1, \dots, 4.$$

We further have an $s - t$ - demand of $q = 2$ (units of flow).

- (a) [5 pts] Prove that the link flow part $\mathbf{x}^*$ of a Wardrop equilibrium flow distribution for this network is uniquely determined
- (b) [6 pts] Show that there are infinitely many possible Wardrop equilibrium path flows in this network.

EXAM - MATHEMATICAL OPTIMIZATION IN TRANSPORT

MSc (M-CEM) FACULTY OF ENGINEERING TECHNOLOGY

Lecturers: M. Uetz, O.A.L. Eikenbroek

Academic year: 2025-2026

2025-202001463-2B

Date: July 1st, 2026

Time: 08:45-11:45 (+ 25% if entitled to use extra time)