

Answer Form

Calculus 2 – Exam – Monday 3 April, 2023

Full name : _____

Student number : _____

▲ ▲ ▲ DO NOT FORGET TO FILL IN YOUR DATA ▲ ▲ ▲

Question 1. Mark your answer with a cross.

Version 1 ☐ A ☐ B ☐ C ☐ D ☐ E ☐ F ☐ G ☒ H

Version 2:



Question 2(a). $\frac{dy}{dx} =$ 3 POSSIBLE WAYS TO EXPRESS

$$\frac{dy}{dx} = \frac{1 - e^x}{1 + \frac{1}{y+1} - 3y^2} \quad \text{OR} \quad \frac{-(e^x - 1)}{1 + \frac{1}{y+1} - 3y^2} \quad \text{OR} \quad \frac{(1 - e^x)(y+1)}{y+2 - 3y^2(y+1)}$$

Question 2(b). $\frac{\partial f}{\partial u}$ at $u = 1$ and $v = 1$ is

$\frac{\partial f}{\partial u} \big _{u=1, v=1} = 96$ Version 1	$\frac{\partial f}{\partial v} \big _{u=1, v=1} = 120$ Version 2
--	---

Question 3(a). $L(x, y) =$

$$L(x, y) = 1 + 1(x-1) - 2(y-1) \quad \text{OR} \quad \text{COULD BE SIMPLIFIED}$$

$$= x - 2y + 2$$

Question 3(b). $P(x, y) =$

$$P(x, y) = 1 + 1(x-1) - 2(y-1) + \frac{1}{2!} (-2(x-1)^2 + 4(y-1)^2)$$

$$\text{OR} \quad \text{COULD BE}$$

$$= 1 + 1(x-1) - 2(y-1) - 1(x-1)^2 + 2(y-1)^2$$

OR COULD BE

$$= 1 + (x-1) - 2(y-1) - (x-1)^2 + 2(y-1)^2$$

OR COULD BE

$$= L(x, y) + \frac{1}{2!} (-2(x-1)^2 + 4(y-1)^2)$$

Question 4. Mark your answer with a cross.

$$\iint_{R_1} y \, dA = \boxed{A} \quad \boxed{B} \quad \boxed{C} \quad \boxed{D} \quad \boxed{E} \quad \boxed{F}$$

$$\iint_{R_2} y \, dA = \boxed{A} \quad \boxed{B} \quad \boxed{C} \quad \boxed{D} \quad \boxed{E} \quad \boxed{F}$$

$$\iint_{R_3} y \, dA = \boxed{A} \quad \boxed{B} \quad \boxed{C} \quad \boxed{D} \quad \boxed{E} \quad \boxed{F}$$

Version 1

F

E

A

Version 2

E

F

B

Question 5(a). $\iint_R f(x,y) \, dA$ as an iterated integral, using the order of integration $dx \, dy$, is

$$\int_0^1 \int_{e^y}^e 4x \, dx \, dy$$

Question 5(b). $\iint_R f(x,y) \, dA =$

$$e^2 + 1$$

Question 5(c). The average value of f over R is

$$e^2 + 1$$

Question 6. The iterated triple integral in spherical coordinates, using the order of integration $d\rho \, d\phi \, d\theta$, is

Version 1

$$\int_0^{\pi/4} \int_0^{\pi/2} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Version 2

$$\int_0^{\pi/4} \int_0^{\pi/2} \int_0^6 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Question 7. Write a full calculation/argumentation clearly in the box.

$$f(x,y) = 2xy^2 - 12x^2 - 6y^2 + 2$$

$$\frac{\partial f}{\partial x} = 2y^2 - 24x = 0 \Rightarrow y^2 - 12x = 0 \Rightarrow y^2 = 12x \quad (1)$$

$$\frac{\partial f}{\partial y} = 4xy - 12y = 0 \Rightarrow \text{From (1)} \quad \left(\frac{y^2}{3}\right) \cdot y - 12y = 0$$

$$\frac{y}{3}(y^2 - 36) = 0$$

Hence $y=0$ or $y^2 - 36 = 0$ i.e. $y = \pm 6$.

For $y=0$, from (1) $x=0$ i.e. $(0,0)$ is a critical pt.

For $y = \pm 6$, from (1) $x=3$ i.e. $(3,6)$ and $(3,-6)$ are critical points

$$\frac{\partial^2 f}{\partial x^2} = -24 \quad \frac{\partial^2 f}{\partial x \partial y} = 4y \quad \frac{\partial^2 f}{\partial y^2} = 4x - 12$$

$$\Delta(0,0) = (-24) \cdot (-12) - 0^2 > 0 \quad \& \quad \frac{\partial^2 f}{\partial x^2}(0,0) = -24 < 0$$

$\Rightarrow f$ has a local maximum at $(0,0)$ & $f(0,0) = 2$.
local maximum value.

$$\Delta(3,6) = (-24) \cdot (0) - (24)^2 < 0 \quad \text{saddle point.}$$

$$\Delta(3,-6) = (-24) \cdot (0) - (-24)^2 < 0 \quad \text{saddle point.}$$

Question 8. Write a full calculation/argumentation clearly in the box.

$$f(x, y, z) = 4x + 2y + 2z$$

constraints are $g(x, y, z) = x - y + z - 1 = 0$ & (4)

$$h(x, y, z) = x^2 + y^2 - 5 = 0 \quad (5)$$

$\nabla f = \lambda \cdot \nabla g + \mu \nabla h$. We obtain the equalities:

$$4 = \lambda + 2\mu \quad (1)$$

$$2 = -\lambda + 2\mu \quad (2)$$

$$2 = \lambda \quad (3)$$

Substitute $\lambda = 2$ to (1) & (2).

From (1) we get: $4 = 2 + 2\mu \Rightarrow x = \frac{1}{\mu}$.

From (2) we get: $2 = -2 + 2\mu \Rightarrow y = \frac{2}{\mu}$.

Substitute $x = \frac{1}{\mu}$ and $y = \frac{2}{\mu}$ into (5). We get:

$$\left(\frac{1}{\mu}\right)^2 + \left(\frac{2}{\mu}\right)^2 = 5 \Rightarrow \mu^2 = 1 \Rightarrow \mu = \pm 1$$

Case 1: $\mu = 1$: Then $x = 1$, $y = 2$. From (4) $z = 2$ $(1, 2, 2)$

Case 2: $\mu = -1$: Then $x = -1$, $y = -2$. From (4) $z = 0$ $(-1, -2, 0)$

$$f(1, 2, 2) = 4 \cdot 1 + 2 \cdot 2 + 2 \cdot 2 = 12 \leftarrow \text{Maximum value obtained at } (1, 2, 2)$$

$$f(-1, -2, 0) = 4 \cdot (-1) + 2 \cdot (-2) + 2 \cdot 0 = -8 \leftarrow \text{Minimum value obtained at } (-1, -2, 0)$$

Question 9(a). Write a full calculation/argumentation clearly in the box.

$$\text{Solve } \begin{cases} u = x + y \\ v = y - \frac{1}{4}x \end{cases} \text{ for } x \text{ and } y$$

$$\begin{cases} u = x + v + \frac{1}{4}x = v + \frac{5}{4}x \\ y = v + \frac{1}{4}x \end{cases} \quad \begin{cases} x = \frac{4}{5}(u - v) = \frac{4}{5}u - \frac{4}{5}v \\ y = v + \frac{1}{5}u - \frac{1}{5}v = \frac{1}{5}u + \frac{4}{5}v \end{cases}$$

$$\begin{aligned} \text{So } J(u, v) &= \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \frac{4}{5} \cdot \frac{4}{5} - \left(-\frac{4}{5}\right) \cdot \frac{1}{5} = \\ &= \frac{16}{25} + \frac{4}{25} = \frac{20}{25} = \frac{4}{5}. \end{aligned}$$

Question 9(b). Write a full calculation/argumentation clearly in the box.

The description of the plate in (u, v) -coordinates is:

$$1 \leq u \leq 3, \quad -4 \leq v \leq 4 \quad (\text{rectangle})$$

$$\text{Density: } \delta(x, y) = \frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}(x + y) = \frac{1}{2}u$$

$$\text{Mass: } \iint_R \delta(x, y) dA = \int_{-4}^4 \int_1^3 \frac{1}{2}u \cdot \frac{4}{5} du dv =$$

$$= \frac{2}{5} \int_{-4}^4 \left[\frac{u^2}{2} \right]_1^3 dv = \frac{2}{5} \cdot 4 \cdot \int_{-4}^4 dv =$$

$$= \frac{2}{5} \cdot 4 \cdot 8 = \frac{64}{5}$$

Question 10(a). Write a full calculation/argumentation clearly in the box.

In cylindrical coordinates, $r = \sqrt{x^2 + y^2}$, so the solid is bounded above by $z = 3\sqrt{1-r}$ and below by $z = \sqrt{1-r}$. The surfaces intersect on $3\sqrt{1-r} = \sqrt{1-r}$ i.e. $8\sqrt{1-r} = 0$ i.e. $8r = 8$ i.e. $r = 1$ (cylinder). So the volume is

$$\int_0^{2\pi} \int_0^1 \int_{\sqrt{1-r}}^{3\sqrt{1-r}} r \, dz \, dr \, d\theta.$$

Question 10(b). Write a full calculation/argumentation clearly in the box.

The solid is symmetric w.r.t the z -axis, so $\bar{x} = \bar{y} = 0$.

We compute $\bar{z} = \frac{M_{xy}}{M}$, where $M = \frac{16}{15}\pi$ is given, and

$$\begin{aligned} M_{xy} &= \iiint_D z \, dV = \int_0^{2\pi} \int_0^1 \int_{\sqrt{1-r}}^{3\sqrt{1-r}} z r \, dz \, dr \, d\theta = \\ &= \int_0^{2\pi} \int_0^1 r \left[\frac{z^2}{2} \right]_{\sqrt{1-r}}^{3\sqrt{1-r}} dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^1 r(8-8r) dr \, d\theta \\ &= 4 \int_0^{2\pi} \int_0^1 (r-r^2) dr \, d\theta = 4 \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{3} \right) d\theta = \\ &= \frac{2}{3} \int_0^{2\pi} d\theta = \frac{2}{3} \cdot 2\pi = \frac{4}{3}\pi. \end{aligned}$$

$$\text{Then } \bar{z} = \frac{\frac{4}{3}\pi}{\frac{16}{15}\pi} = \frac{5}{4}.$$

The center of mass is then $(0, 0, \frac{5}{4})$.