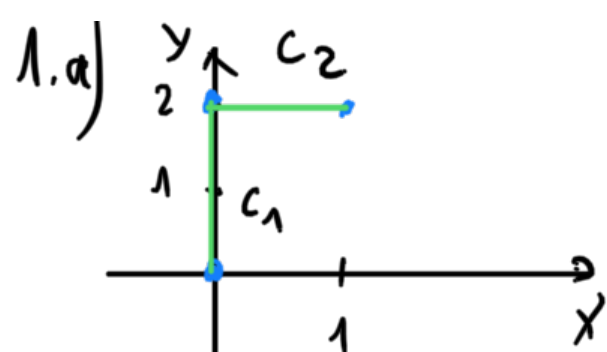




$$C_1: x=0, 0 \leq y \leq 2 \Rightarrow r_1(t) = t\mathbf{j}, 0 \leq t \leq 2$$

$$C_2: 0 \leq x \leq 1, y=2 \Rightarrow r_2(t) = t\mathbf{i} + 2\mathbf{j}, 0 \leq t \leq 1$$

additivity of line integrals: $I = I_1 + I_2 = 0 + \frac{13}{3} = \frac{13}{3}$ (1) for additivity

with $I_1 = \int_{C_1} (x^2 + y^2) dx + 2xy dy = \int_0^2 0 dt = 0$
 $dx=0, dy=dt$
 $2xy=0$

$$I_2 = \int_{C_2} (x^2 + y^2) dx + 2xy dy = \int_0^1 t^2 + 4 dt = \frac{13}{3}$$

$$dx=dt, dy=0$$

$$x^2 + y^2 = t^2 + 4$$

1.b) path independence of $\int_C \mathbf{F} \cdot d\mathbf{r} \Leftrightarrow \nabla \times \mathbf{F} = \mathbf{0} \Leftrightarrow \mathbf{r} \cdot (\nabla \times \mathbf{F}) = 0$ (9.5)
 $\mathbf{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

the last identity is true, because $\frac{\partial N}{\partial x} = 2y = \frac{\partial M}{\partial y}$ (9.5)

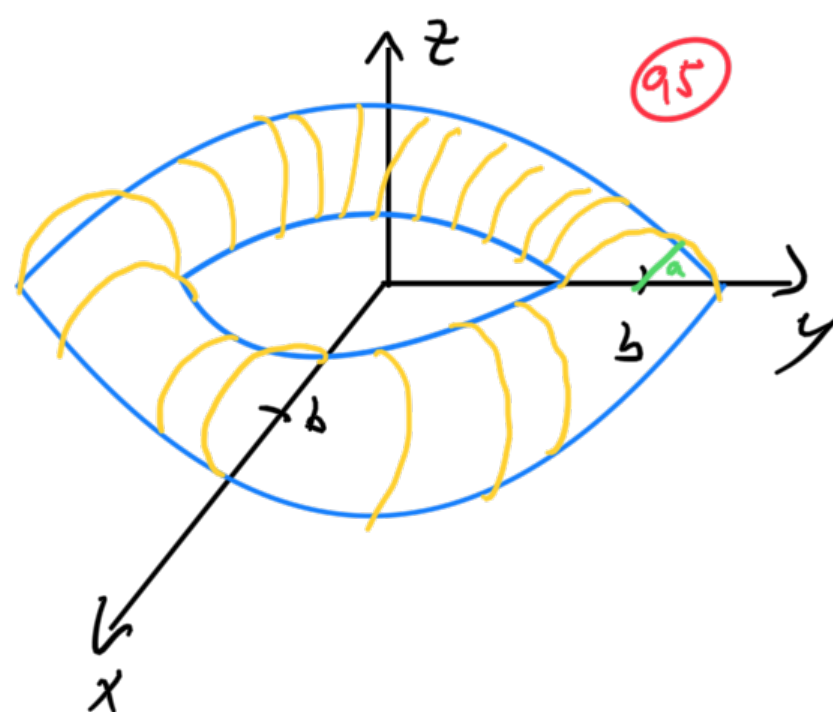
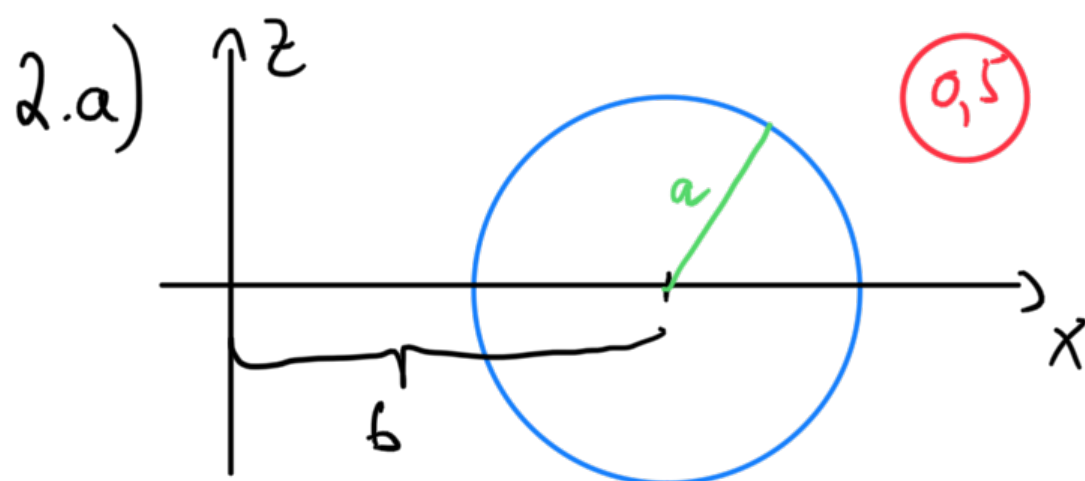
1.c) outward flux (1) $= \oint_C \mathbf{F} \cdot \mathbf{n} ds \stackrel{\text{Green}}{=} \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$ (1)

$$= \int_0^1 \int_{2x^2}^{2x} (4x) dy dx = \int_0^1 \underbrace{4x(2x - 2x^2)}_{= 8x^2 - 8x^3} dx$$

use given parametrization

$$= \int_0^1 8x^2 - 8x^3 dx = \left. \frac{8}{3}x^3 - 2x^4 \right|_0^1 = \frac{8}{3} - 2 = \frac{2}{3}$$

total: $3 + 1 + 3 = 7$



2.b) point on circle in xz plane : $(b + a \cos u)i + (a \sin u)k$ (0,5)
 $0 \leq u \leq 2\pi$
 rotation of a point $c, d, e \in \mathbb{R}$ around z -axis

$$c \cos v i + d \sin v j + e k, \quad 0 \leq v \leq 2\pi \quad (0,5)$$

⇒ parametrization of torus:

$$r(u, v) = (b + a \cos u) \cos v i + (b + a \cos u) \sin v j + a \sin u k \quad (0,5)$$

$0 \leq u, v \leq 2\pi$

(note: the term $b + a \cos u$ in the j -component, because we need a circle in the yz -plane as well)

if parametrization is correct → all points for b.)

2.c)

$$\text{area of } T = \iint_T d\sigma = \int_0^{2\pi} \int_0^{2\pi} |r_u \times r_v| du dv = (*) \quad (0,5)$$

$$r_u = -a \sin u \cos v i - a \sin u \sin v j + a \cos u k$$

$$r_v = -(b + a \cos u) \sin v i + (b + a \cos u) \cos v j \quad (0,5)$$

$$r_u \times r_v = \det \begin{pmatrix} i & j & k \\ -a \sin u \cos v & -a \sin u \sin v & a \cos u \\ -(b + a \cos u) \sin v & (b + a \cos u) \cos v & 0 \end{pmatrix}$$

$$= -(\underline{a(b + a \cos u) \cos u \cos v}) i - (\underline{a(b + a \cos u) \cos u \sin v}) j$$

$$- (\underline{a(b + a \cos u) \sin u (\cos v)^2 + a(b + a \cos u) \sin u (\sin v)^2}) k$$

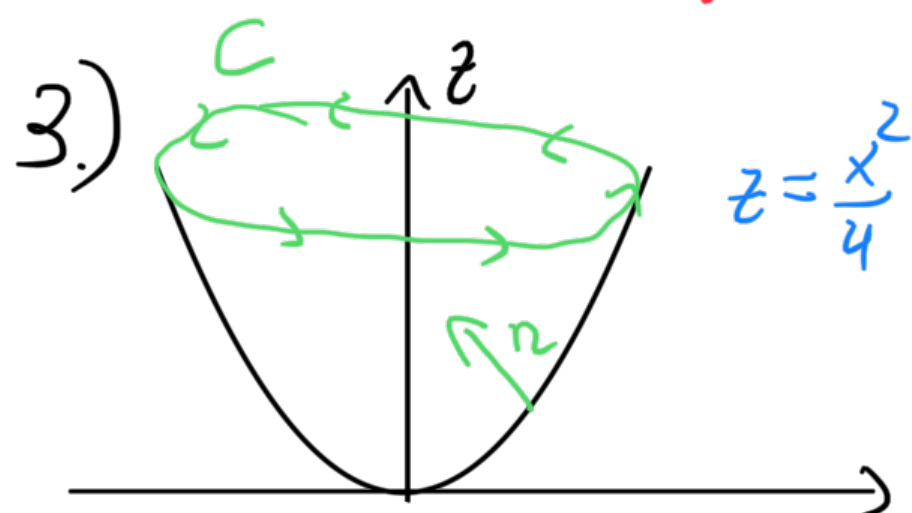
$$= \underline{a(b + a \cos u) \sin u} \quad (0,5)$$

$$\Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = \underline{a(b + a \cos u)} \quad (0,5)$$

$$\Rightarrow (*) = \int_0^{2\pi} \int_0^{2\pi} a(b + a \cos u) \, du \, dv = 4\pi^2 ab + 2\pi a \underbrace{\int_0^{2\pi} \cos u \, du}_{= \sin u \Big|_0^{2\pi} = 0} \quad (0,5)$$

$$= 4\pi^2 ab$$

$$\text{total: } 1 + 1,5 + 3 = 5,5$$



C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in the $z=1$ plane.

circulation of $\mathbf{F} = \oint_C \mathbf{F} \cdot d\mathbf{s} \stackrel{(0,5)}{=} \stackrel{\text{Stokes}}{=} \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA = (*) \quad (1)$

(i) compute normal $\mathbf{n}$: S is part of levelset $f(x,y,z) = \frac{x^2}{4} + \frac{y^2}{9} - z = 0 \quad (0,5)$

$$\Rightarrow \mathbf{n} \stackrel{(0,5)}{=} \pm \frac{\nabla f}{|\nabla f|}, \quad \nabla f = \frac{x}{2}\mathbf{i} + \frac{2y}{9}\mathbf{j} - \mathbf{k} \quad (0,5)$$

right-hand rule (see sketch): $\mathbf{n}$ has positive k -component 1

$$\Rightarrow \mathbf{n} = -\frac{\nabla f}{|\nabla f|}$$

$$(ii) \nabla \times \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_{xz} + h_y & 2x + y^2 & -2y^2 + 2z \end{pmatrix} = -4y\mathbf{i} + 9x\mathbf{j} \quad (0,5)$$

$$(iii) \Rightarrow (\nabla \times \mathbf{F}) \cdot \mathbf{n} = \frac{-1}{|\nabla f|} \underbrace{(-4y\mathbf{i} + 9x\mathbf{j}) \cdot \left(\frac{x}{2}\mathbf{i} + \frac{2y}{9}\mathbf{j} - \mathbf{k}\right)}_{= -2xy + 2xy = 0} = 0 \quad (0,5)$$

$$\Rightarrow (*) = 0. \quad (0,5)$$

alternative for (i): (i') parametrize S : cylinder coordinates

$$\left. \begin{aligned} x &= 2r \cos t \\ y &= 3r \sin t \\ z &= r^2 \end{aligned} \right\} \Rightarrow \vec{r}(r, t) = 2r \cos t \mathbf{i} + 3r \sin t \mathbf{j} + r^2 \mathbf{k} \quad \left. \begin{aligned} 0 \leq t \leq 2\pi, \\ 0 \leq r \leq 1. \end{aligned} \right\} \quad 0,5$$

$$\frac{x^2}{4} + \frac{y^2}{9} = z$$

$$\eta = \frac{\vec{r}_r \times \vec{r}_t}{|\vec{r}_r \times \vec{r}_t|}, \quad \vec{r}_r = 2 \cos t \mathbf{i} + 3 \sin t \mathbf{j} + 2r \mathbf{k}$$

$$\vec{r}_t = -2r \sin t \mathbf{i} + 3r \cos t \mathbf{j}$$

$$\Rightarrow \vec{r}_r \times \vec{r}_t = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 \cos t & 3 \sin t & 2r \\ -2r \sin t & 3r \cos t & 0 \end{pmatrix} = -6r^2 \cos t \mathbf{i} - 4r^2 \sin t \mathbf{j} + 6r \mathbf{k} \quad 0,5$$

right-hand rule $\eta = \frac{\vec{r}_r \times \vec{r}_t}{|\vec{r}_r \times \vec{r}_t|} \quad 1$

(iii') $\nabla_x T \cdot \eta = \frac{1}{|\vec{r}_r \times \vec{r}_t|} (-12r \sin t \mathbf{i} + 18r \cos t \mathbf{j}) \cdot (-6r^2 \cos t \mathbf{i} - 4r^2 \sin t \mathbf{j} + 6r \mathbf{k})$

$$= 0. \quad 0,5$$