

1. (15 points) Consider the minimization problem

$$\min z(x_1, x_2) = x_1^2 + 2x_2^2 + 2x_1x_2 - 2x_1 - 4x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\geq 3 \\ x_1 &\geq 0. \end{aligned}$$

- (a) Compute the unconstrained minimum for z , and verify that it is infeasible.
 - (b) Write out the KKT conditions, and solve the system such as to find a (constrained) local minimum $\mathbf{x}^*$.
 - (c) Prove that your solution $\mathbf{x}^*$ the unique (constrained) minimizer of the problem.
2. (10 points) Recall the steepest descent algorithm for minimization of a function $z : \Re^n \rightarrow \Re$, and recall that, in each iteration, we first find the direction of steepest (feasible) descent $\mathbf{d}$, and then find the optimal step size α in direction $\mathbf{d}$.

Show that, if $z : \Re^n \rightarrow \Re$ is a convex function of $\mathbf{x}$, then also $f : \Re \rightarrow \Re$ is a convex function, where f is defined by

$$f(\alpha) = z(\mathbf{x} + \alpha\mathbf{d}).$$

Why is this important for the steepest descent method?

3. (10 points) Consider the unconstrained minimization problem

$$\min z(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 2x_1x_2 - 2x_1 - 4x_2$$

- (a) Perform the first iteration of the steepest descent method, starting at point $\mathbf{x}^0 = (0, 0, 0)$. What is the steepest descent direction $\mathbf{d}^0$ and what is the optimal step length α_0 ? (The latter can be computed analytically.)
- (b) Compute the steepest descent direction $\mathbf{d}^1$ of the next iteration and verify that the first two descent directions are orthogonal.

4. (15 points) Consider the problem to minimize

$$z : \Re^2 \rightarrow \Re \quad z(\mathbf{x}) = x_1^2 + 2x_2^2 - 2x_1x_2 - 10x_2,$$

subject to

$$\begin{aligned} x_1 - x_2 &\geq -5, \\ -4x_1 + x_2 &\geq -20, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Explain the first iteration of the convex combinations (Frank-Wolfe) method on this problem, when starting at point $\mathbf{x}^0 = (0, 0)$:

- (a) Plot the feasible region of the linearized problem, write down the corresponding LP, and solve it graphically to determine the direction $\mathbf{d}^0$. Also compute the step length α_0 , and determine the next point $\mathbf{x}^1 = \mathbf{x}^0 + \alpha_0 \mathbf{d}^0$.
- (b) Is it true that any step length $\alpha_0 \in [0, 1]$ results in a decrease? Give a condition under which this is the case.