

Course : Calculus 2

Module : 2B

Course code : 2020012::[19|20|21|22]

Date : Friday 14 June, 2024

Time : 13:45 – 15:45 (2h)

Reference : CAL2-2324-Q4-EXAM

## Calculus 2

### Exam

### Instructions

This exam contains 10 exercises. You shall use the provided *answer form* to submit your answers.

► For exercises 1–6, you are only required to fill in the **final answer** on the answer form.

► For exercises 7–10, you are required to write down a **full calculation and argumentation**.

You will hand in your answer form only. **Any text outside the answer form will not be considered.**

If you run out of space, you can use the extra space at the end of the answer form. Refer clearly to that space in the original answer.

Do not write with red pen or pencil. Do not use correction fluid or tape.

**▲ THE USE OF ELECTRONIC DEVICES IS NOT ALLOWED. ▲**

### Final answer questions

**Report all your answers on the answer form.**

1. Use implicit differentiation to compute  $\frac{dy}{dx}$  at the point  $(x, y) = (0, 4)$ , if [2 pt]

$$y \sin(x) = y^2 - 20x - 16.$$

2. Consider the function  $f(x, y) = x^4 \cos(y) - 3xy^2$ .

Let  $P_2(x, y)$  be the second-order Taylor polynomial generated by  $f$  at the point  $(1, \pi)$ .

(a) Determine the coefficient of  $xy$  in the expansion of  $P_2(x, y)$ . [2 pt]

(b) Compute  $P_2(1, \pi)$ . [1 pt]

3. Of a two-variable function  $f(x, y)$ , it is known that  $\nabla f(x, y) = (x^3 - 2x)\mathbf{i} + (y^3 - 2y)\mathbf{j}$ . [3 pt]

For each of the following critical points of  $f$ :

$$(0, -\sqrt{2}), \quad (0, 0), \quad (\sqrt{2}, -\sqrt{2}),$$

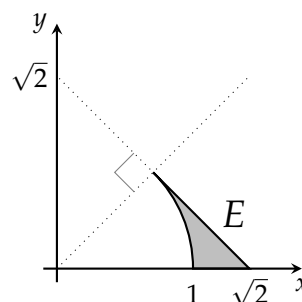
determine whether  $f$  attains a local maximum or a local minimum at it, or whether  $f$  has a saddle point at it.

4. The region  $E$  of the Cartesian plane, shaded in the figure, is bounded by the lines  $y = 0$  and  $x + y = \sqrt{2}$ , and by the circle of radius 1 centered at the origin. Express [3 pt]

$$\iint_E x^2 \, dx \, dy$$

as an iterated double integral in polar coordinates, using the order of integration  $dr \, d\theta$ .

Do not evaluate the integral.



5. Consider a point  $P$  in the three-dimensional space. In spherical coordinates,  $P$  lies at the intersection of the sphere  $\rho = \frac{\pi}{2}$  with the cone  $\phi = \frac{\pi}{2} + \frac{1}{100}$  and the half-plane  $\theta = \frac{\pi}{2} - \frac{1}{50}$ . [3 pt]

For each of the Cartesian coordinates  $(x, y, z)$  of the point  $P$ , determine whether it is positive, negative or zero.

6. Consider the following transformation of the first quadrant of the Cartesian plane: [2 pt]

$$\begin{cases} x = 2u e^v \\ y = 3e^{-v} \end{cases} \quad \text{or, equivalently,} \quad \begin{cases} u = \frac{xy}{6} \\ v = \ln(3) - \ln(y) \end{cases}.$$

The transformation maps a region  $G$  of area 1 of the  $(u, v)$ -plane into a region  $R$  of the  $(x, y)$ -plane. Find the area of the region  $R$ .

## Open questions

**The full solutions to exercises 7—10 must be clearly written down on the answer form, including calculations and argumentations.**

Points will not be awarded for reaching a correct result if this is not supported by a correct procedure and by a sound and clear argumentation.

7. (a) Use the method of Lagrange multipliers to determine the minimum value of the function  $f(x, y) = x^3 - 4y^2 - 1$  on the ellipse  $x^2 + 2y^2 = 1$ . [4 pt]

Clearly indicate the point (or points) where the minimum value is attained.

- (b) The maximum value of  $f$  on the ellipse is 0, attained at the point  $(1, 0)$ . [1 pt]

Consider the level curve  $f(x, y) = 3$ . Does it intersect the given ellipse? If so, specify at which points. If not, briefly explain why.

("To intersect" means "to have points in common".)

8. The region  $E$  in the Cartesian space is bounded above by the paraboloid  $z = x^2 + y^2 + 4$  and below by the paraboloid  $z = 2x^2 + 2y^2$ . [5 pt]

Compute the average value of  $f(x, y, z) = x^2 + y^2$  over  $E$ .

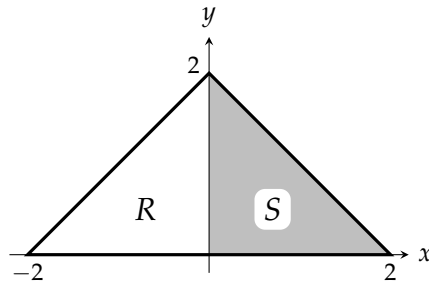
9. The function  $f(x, y) = x^2 - y^2 + e^{1-x^2-y^2}$  has gradient [5 pt]

$$\nabla f(x, y) = 2x(1 - e^{1-x^2-y^2}) \mathbf{i} - 2y(1 + e^{1-x^2-y^2}) \mathbf{j}.$$

Find the absolute maximum and minimum values of  $f$  on the closed region  $R$  bounded by the circle  $x^2 + y^2 = 4$ . Clearly indicate at which points those values are attained.

You do not need to approximate the result, but it is good to remember that  $e \approx 2.7$ .

10. The figure below shows a thin triangular lamina  $T$  on the Cartesian plane bounded by the lines  $y = 0$ ,  $y = x + 2$  and  $y = -x + 2$ . The lamina is composed of two parts  $R$  and  $S$  of different materials: the region  $R$  (white in the figure) is the part of the lamina on the second quadrant, while the remaining region  $S$  (shaded) is the part lying on the first quadrant. [5 pt]



The density of the lamina  $T$  is given by the following density function:

$$\rho(x, y) = \begin{cases} 1 & \text{on the region } R, \\ 1 + y & \text{on the region } S. \end{cases}$$

Furthermore, it is known that the total mass of  $T$  is  $\frac{16}{3}$ .

The center of mass of  $T$  is  $(\bar{x}, \frac{3}{4})$ . Compute  $\bar{x}$ .





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