

Q1.a) $\mathbf{r}(t) = a \cos(t)\mathbf{i} + a \sin(t)\mathbf{j} + a \cos(t)\mathbf{k}, \quad 0 \leq t \leq \pi/2$

b) $\mathbf{r}'(t) = -a \sin(t)\mathbf{i} + a \cos(t)\mathbf{j} - a \sin(t)\mathbf{k}$

$$|\mathbf{r}'(t)| = a \sqrt{\cos^2(t) + \sin^2(t)} = a \sqrt{1 + \sin^2(t)}$$

$$\begin{aligned} \text{mass} &= \int_C \rho(x, y, z) \, ds \\ &= \int_0^{\pi/2} a^2 \cos(t) \sin(t) \cdot a \sqrt{1 + \sin^2(t)} \, dt \\ &= \frac{a^3}{2} \int_1^2 \sqrt{u} \, du = \frac{a^3}{3} (2\sqrt{2} - 1) \end{aligned}$$

$u = 1 + \sin^2(t)$
 $du = 2 \sin(t) \cos(t) dt$

Q2)(a) parametrization: $\mathbf{r}(t) = (2t)\mathbf{i} + \mathbf{j} + (1+t)\mathbf{k}, \quad 0 \leq t \leq 1$

$$\Rightarrow dx = 2dt, \, dy = 0, \, dz = dt$$

$$\begin{aligned} &\Rightarrow \int_C 2x e^{(z-1)^2} dx + 2y^2 z \, dy + y^3 \, dz \\ &= \int_0^1 8t e^{t^2} + 1 \, dt \\ &= \left[4e^{t^2} + t \right]_0^1 = 4e - 3 \end{aligned}$$

(b) No, because the integral equals $\int_C \mathbf{F} \cdot d\mathbf{r}$ with $\mathbf{F}(x, y, z) = 2x e^{(z-1)^2} \mathbf{i} + 2y^2 z \mathbf{j} + y^3 \mathbf{k}$

and $\mathbf{F}$ cannot have a potential by the component test:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x e^{(z-1)^2} & 2y^2 z & y^3 \end{vmatrix} = (3y^2 - 2y^2)\mathbf{i} - (0 - 4x(z-1)e^{(z-1)^2})\mathbf{j} + (0-0)\mathbf{k} \neq \mathbf{0}$$

Q3) circulation = $\oint_C \mathbf{F} \cdot d\mathbf{s}$

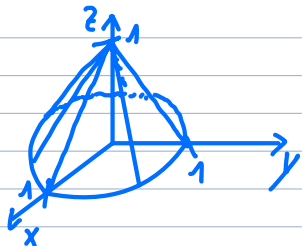
$$\stackrel{\text{Green}}{=} \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \int_0^2 \int_0^{1-\frac{x}{2}} (-2y) dy dx$$

$$R: 0 \leq x \leq 2, 0 \leq y \leq 1 - \frac{x}{2}$$

$$= - \int_0^2 \left(1 - \frac{x}{2}\right)^2 dx = -\frac{2}{3}$$

Q4.a)



$$b) \text{ flux}_A = \iint_A \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

$$\vec{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + (1-r) \mathbf{k}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & -1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r \mathbf{k}$$

normal has pos. k-comp. : $\mathbf{n} = + \vec{r}_r \times \vec{r}_\theta / |\vec{r}_r \times \vec{r}_\theta|$

$$\Rightarrow \iint_A \mathbf{F} \cdot \mathbf{n} \, d\sigma = + \int_0^1 \int_0^{2\pi} \underbrace{\mathbf{F}(r, \theta)}_{= r \sin \theta \mathbf{i} + r^2 \mathbf{k}} \cdot (\vec{r}_r \times \vec{r}_\theta) \, d\theta \, dr$$

$$= + \int_0^1 \int_0^{2\pi} r^2 \cos \theta \sin \theta + r^3 \, d\theta \, dr$$

$$= 0 + \frac{\pi}{2}$$

$$c) \text{ flux}_S = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \underbrace{\text{div}(\mathbf{F})}_{2(z-1)} \, dV$$

$$= \int_0^1 \int_0^{2\pi} \int_0^{1-r} 2(z-1) \, r \, dz \, d\theta \, dr$$

$D: (x, y, z): x = r \cos \theta, y = r \sin \theta, z = z, \quad 0 \leq z \leq 1-r, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$

$$= 2\pi \int_0^1 r^3 - r \, dr = -\frac{\pi}{2}$$

$$d) \text{ influx} = \iint_B \mathbf{F} \cdot \mathbf{n}_{in} \, d\sigma = - \iint_B \mathbf{F} \cdot \mathbf{n}_{out} \, d\sigma$$

$$= \iint_A \mathbf{F} \cdot \mathbf{n}_{out} \, d\sigma - \iint_S \mathbf{F} \cdot \mathbf{n}_{out} \, d\sigma = b - c$$