

Grading scheme: Exam Vector Calculus BMT, CE, CSE, ME, Oct 8th 2024.

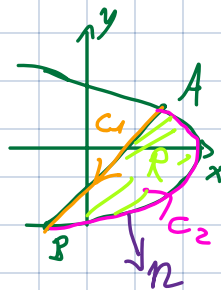
1. (a) $C_1: r(t) = A + tB = (2-3t)i + (1-3t)j, \quad 0 \leq t \leq 1$
 $r'(t) = -3i - 3j$

$$\begin{aligned} \text{work} &= \int_{C_1} F \cdot T ds = \int_0^1 F(r(t)) \cdot r'(t) dt \\ &= \int_0^1 \underbrace{[(2-3t+1-3t)i - (2-3t+1-3t)j]}_{= (3-6t)i - (3-6t)j} \cdot [-3i - 3j] dt \\ &= \underbrace{\int_0^1 [-3(3-6t) + 3(3-6t)] dt}_{= 0} \\ &= \underline{\underline{0}} \end{aligned}$$

(b) $C_2: r(t) = (3-t^2)i + tj, \quad -2 \leq t \leq 1$
 $r'(t) = -2ti + j$

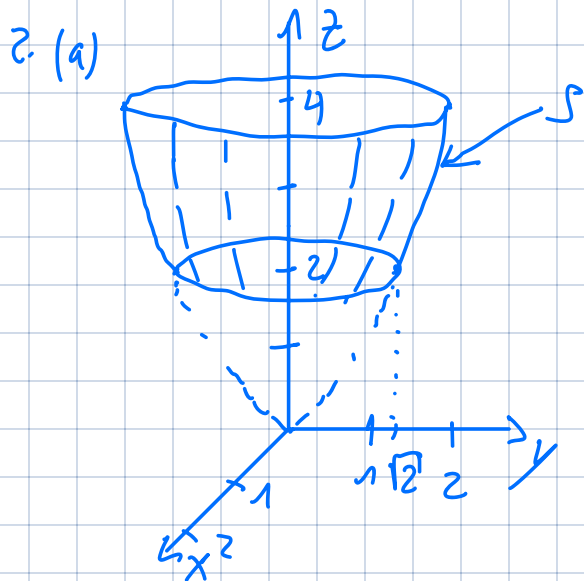
$$\begin{aligned} \text{flow} &= \int_{C_2} F \cdot T ds = \int_{-2}^1 F(r(t)) \cdot r'(t) dt \\ &= \int_{-2}^1 \underbrace{[(3-t^2+t)i - (3-t^2+t)j]}_{= -6t + 2t^3 - 2t^2 - 3 + t^2 - t} \cdot [-2ti + j] dt \\ &= \int_{-2}^1 [2t^4 - \frac{2}{3}t^3 - \frac{7}{2}t^2 - 3t] dt = \dots = \underline{\underline{-9}} \end{aligned}$$

(c) $\text{flux} = \int_C F \cdot n ds \stackrel{\text{Green}}{=} \int_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} dA$
 $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 1 - 1 = 0$
 (outer normal) $\leftarrow$ region enclosed by C



remark: Using additivity of line integrals, one also has

$$\begin{aligned} \int_C F \cdot n ds &= \int_C F \cdot n ds + \int_C F \cdot n ds = (i) + (ii) = 0 \\ (i) &= \int_0^1 M(r(t))y' - N(r(t))x' dt = -3 \int_0^1 M(r(t)) - N(r(t)) dt = 0 \\ (ii) &= \int_{-2}^1 M(r(t))y' - N(r(t))x' dt = \int_{-2}^1 3 - 5t - 3t^2 + 2t^3 dt = 0 \end{aligned}$$



(b) parametrization:

$$\vec{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r^2 \mathbf{k}$$

$$\text{with } \sqrt{2} \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

for surface area element $ds = |\vec{r}_r \times \vec{r}_\theta| dr d\theta$:

$$\vec{r}_r(r, \theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + 2r \mathbf{k}$$

$$\vec{r}_\theta(r, \theta) = -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j}$$

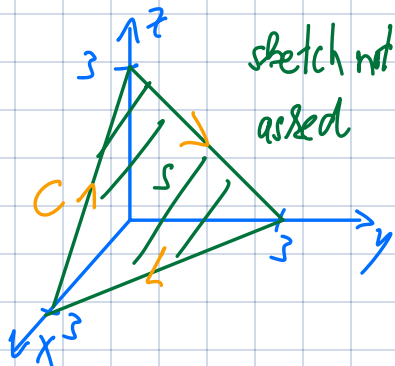
$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = -2r^2 \cos \theta \mathbf{i} - 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}$$

$$\Rightarrow |\vec{r}_r \times \vec{r}_\theta| = \sqrt{4r^4 + r^2} = r \sqrt{1 + 4r^2}$$

$$\Rightarrow \text{area of } S = \iint_S ds = \int_{\sqrt{2}}^2 \int_0^{2\pi} r \sqrt{1 + 4r^2} d\theta dr$$

$$\stackrel{\substack{u = 1 + 4r^2 \\ du = 8r}}{=} 2\pi \frac{1}{8} \int_9^{17} \sqrt{u} du = \frac{\pi}{4} \frac{2}{3} u^{3/2} \Big|_9^{17} = \frac{\pi}{6} [17^{3/2} - 27]$$

3. (a) $r(x,y) = xi + yj + (3-x-y)k$
with $0 \leq x \leq 3$ and $0 \leq y \leq 3-x$



(b) $\text{flux} = \iint_S F \cdot n \, d\sigma = \pm \int_0^3 \int_0^{3-x} F(x,y,3-x-y) \cdot (r_x \times r_y) \, dy \, dx$

it is $r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = 1i + 1j + 1k$

positive, hence $n = -\frac{r_x \times r_y}{|r_x \times r_y|}$

remark: $n = \pm \frac{1}{\sqrt{3}}(1i + 1j + 1k)$
could be seen directly

and $F(x,y,3-x-y) = xi + (3-x-y)j - xyk$

$F(x,y,3-x-y) \cdot r_x \times r_y = 3 - y - xy = 3 - (1+x)y$

$\Rightarrow \text{flux} = \int_0^3 \int_0^{3-x} (3 - (1+x)y) \, dy \, dx = \int_0^3 \left(3y - \frac{(1+x)y^2}{2} \right) \Big|_0^{3-x} \, dx$
 $= \int_0^3 \left(\frac{1}{2}(9 - 6x + x^2) - \frac{1}{2}(1+x)(3-x)^2 \right) \, dx$
 $= \int_0^3 \left(\frac{1}{2}x^3 - \frac{5}{2}x^2 + \frac{9}{2}x - \frac{9}{2} \right) \, dx = \left[\frac{1}{8}x^4 - \frac{5}{6}x^3 + \frac{9}{4}x^2 - \frac{9}{2}x \right]_0^3 = \frac{243}{8} - 36 = -5.625 = -\frac{45}{8}$

(c) circulation of $F = \oint_C F \cdot T \, ds \stackrel{\text{Stokes}}{=} \iint_S (\nabla \times F) \cdot n \, d\sigma = (*)$

with $\nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x & z & -xy \end{vmatrix} = (-x-1)i - (-y)j + 0k$

and n needs to have a negative k -component (right-hand rule).

$\Rightarrow$ (using b) $n = -\frac{1}{\sqrt{3}}(1i + 1j + 1k)$

and $F \cdot n = -\frac{1}{\sqrt{3}}(-1-x+y) = \frac{1+x-y}{\sqrt{3}}$ and $d\sigma = \sqrt{3} \, dy \, dx$

$\Rightarrow (*) = \int_0^3 \int_0^{3-x} (1+x-y) \, dy \, dx = \int_0^3 \left((1+x)y - \frac{y^2}{2} \right) \Big|_0^{3-x} \, dx$
 $= \int_0^3 \left(-\frac{3}{2} + 5x - \frac{3}{2}x^2 \right) \, dx = -\frac{9}{2} + \frac{45}{2} - \frac{27}{2} = \frac{9}{2}$