

LINEAR ALGEBRA_A Date : April 21, 2023
Time : 13.45 – 15.45 hrs

First read these instructions carefully:

This test consists of 10 exercises.

For exercises 1, 4, and 6 you must only write down the final answers on the answer sheet. The complete solutions for the remaining exercises must be accurately written down on the answer sheet, including calculations and argumentation. The use of dictionaries or electronic devices is not allowed. The answer sheet must be handed in. The grade is calculated by dividing the total number of points by 4, then adding 1.

1. (3pts.) *This is a final answer question. Provide your final answer (and only your final answer) on the supplied space on the answer sheet.*

The matrix given is the augmented matrix for a system of linear equations. Give the parametric vector form for the general solution of this system of linear equations.

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right)$$

2. (3pts.) *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

$$\text{Let } A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Calculate the following expression: $AB^T + \mathbf{v}\mathbf{v}^T$

3. (3pts.) *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

Given $N = \begin{pmatrix} 7 & 1 \\ 2 & 8 \end{pmatrix}$. Find a regular (invertible) 2×2 matrix M such that $M^3 = M^2N - 5M^2$.

4. (3pts.) *This is a final answer question. Provide your final answer (and only your final answer) on the supplied space on the answer sheet.*

Determine which of the following set(s) form a basis of $\mathbb{R}^3$. *Only one option is correct.*

$$(A) \ S_1 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$$

$$(C) \ S_3 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 5 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix} \right\}$$

$$(B) \ S_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \right\}$$

$$(D) \ S_4 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$$

5. (4pts.) *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

Let $C \in \mathbb{R}^{3 \times 3}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be linearly independent vectors in $\mathbb{R}^3$.

$$\text{We are given: } C\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad C\mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad C\mathbf{z} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Find the value of the determinant of the matrix C .

6. *This is a final answer question. Provide your final answer (and only your final answer) on the supplied space on the answer sheet.*

The matrix D is given as $D = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$.

- a) (2pts.) Determine the eigenvalues of D .
- b) (2pts.) Determine the corresponding eigenspaces of D .

7. *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

Consider two planes p_1 and p_2 in $\mathbb{R}^3$ given as below.

$$p_1: 2x_1 + 3x_2 - 2x_3 + 2 = 0$$

$$p_2: \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

- a) (1pt.) Provide an equation for p_2 similar to the equation for p_1 .
- b) (3pts.) Find the intersection of these two planes.

8. *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be a linear transformation given with:

$$T(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 + x_3 \\ 2x_1 + x_2 - 3x_3 \\ x_1 - 3x_2 - 2x_3 \\ x_1 + 2x_2 - 5x_3 \end{pmatrix}$$

- a) (2pts.) Let A be the associated representation matrix for T , determine A .
- b) (3pts.) Is T surjective (onto) and/or injective (one-to-one)? Explain your answer, providing all relevant arguments.

9. (3pts.) *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

Consider the linear transformation P which rotates every vector in $\mathbb{R}^2$ about the origin through an angle of $\frac{\pi}{4}$ radians (counterclockwise).

Determine the representation matrix of P by examining the images of $\mathbf{e}_1$ and $\mathbf{e}_2$.

10. (4pts.) *Use the supplied space on the answer sheet and include clear argumentation and calculation. All your work will be graded.*

Let A and $B \in \mathbb{R}^{n \times n}$. Suppose that the matrices A and B have a common eigenvector which is $\mathbf{v}$. Show that $\det(AB - BA) = 0$.

Do not remove the staple!

Exercises 1, 4, 6 for versions B, C, D are on page 6

Linear Algebra April 03, 2023

Answer sheet Answer the questions legibly in the spaces provided. If you run out of room for an answer, continue on the back of the answer sheet by clearly referring to the question.

Solutions

Name and Student Number: _____

Programme: _____

Question	1	2	3	4	5	6	7	8	9	10	Total
Points Possible	3	3	3	3	4	2+2	1+3	2+3	3	4	36
Points Earned											

1. Only provide your final answer.

The parametric vector form for the general solution is:
 $x_3 \begin{pmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}; x_3, x_5 \in \mathbb{R}$ OR $\text{Span} \left\{ \begin{pmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ OR other equivalent forms.

2. Show all your work.

$$AB^T = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ -1 & -3 \end{pmatrix}$$

$$vv^T = \begin{pmatrix} 1 \\ 2 \end{pmatrix} (1 \ 2) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$AB^T + vv^T = \begin{pmatrix} -2 & 0 \\ -1 & -3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$$

3. Show all your work.

Given $M^3 = M^2N - 5M^2$, if M is invertible then M^{-1} exists.
 $\Rightarrow M^{-1}M^{-1}M^3 = M^{-1}M^{-1}M^2N - 5M^{-1}M^{-1}M^2$! multiply both sides with $M^{-1}M^{-1}$ from the left
 $\Rightarrow M = N - 5I$

$$\Rightarrow M = \begin{pmatrix} 7 & 1 \\ 2 & 8 \end{pmatrix} - 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 2 & 8 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$$

M is invertible with $M^{-1} = \begin{pmatrix} 3/4 & -1/4 \\ -1/2 & 1/2 \end{pmatrix}$

4. Add a check mark (✓) to the correct option.

☐ (A)

☒ (B)

☐ (C)

☐ (D)

5. Show all your work.

Let $D \in \mathbb{R}^{3 \times 3}$ and $D = (\vec{x} \ \vec{y} \ \vec{z})$

Then we have : $CD = (C\vec{x} \ C\vec{y} \ C\vec{z}) = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Then we have :

$$\det C \det D = \det CD = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

rows are equal

Since $\vec{x}, \vec{y}, \vec{z}$ are given as linearly independent vectors,

$D = (\vec{x} \ \vec{y} \ \vec{z})$ is an invertible matrix,

therefore, $\det D \neq 0$,

$$\det CD = 0 \wedge \det D \neq 0 \Rightarrow \det C = 0.$$

6. a) Only provide your final answer.

$$\lambda_1 = -1 + 2i$$

$$\lambda_2 = -1 - 2i$$

b) Only provide your final answer.

$$E_{\lambda_1} = \text{Span} \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$$

$$E_{\lambda_2} = \text{Span} \left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$$

A

7. a) Show all your work.

$$P_2: \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}; \alpha, \beta \in \mathbb{R}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in P_2 \Rightarrow \left. \begin{array}{l} x_1 = 1 - 2\alpha - \beta \\ x_2 = \alpha \\ x_3 = \beta \end{array} \right\}$$

we have, $x_1 = 1 - 2x_2 - x_3$

b) Show all your work.

We must solve the two equations for p_1 & p_2 :

$$p_1: 2x_1 + 3x_2 - 2x_3 = -2$$

$$p_2: x_1 + 2x_2 + x_3 = 1$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & -2 \\ 1 & 2 & 1 & 1 \end{array} \right) \sim \dots \sim \left(\begin{array}{ccc|c} 1 & 0 & -7 & -7 \\ 0 & 1 & 4 & +4 \end{array} \right)$$

$$\Rightarrow x_1 = -7 + 7x_3$$

$$x_2 = +4 - 4x_3$$

$$x_3 = x_3$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -7 + 7x_3 \\ +4 - 4x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -7 \\ +4 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ -4 \\ 1 \end{pmatrix}$$

$$L: \begin{pmatrix} -7 \\ +4 \\ 0 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} 7 \\ -4 \\ 1 \end{pmatrix} \right\}$$

8. a) Show all your work.

$$T(\vec{e}_1) = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \quad T(\vec{e}_2) = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 2 \end{pmatrix} \quad T(\vec{e}_3) = \begin{pmatrix} 1 \\ -3 \\ -2 \\ -5 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ 1 & -3 & -2 \\ 1 & 2 & -5 \end{pmatrix}$$

b) Show all your work.

$$A \sim \dots \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

injective $\Rightarrow \text{Null } A = \{\vec{0}\}$ therefore
A is injective.

surjective $\Rightarrow T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$
 $\text{col } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \neq \mathbb{R}^4$
A is not surjective.

9. Show all your work.

the representation matrix of P is

$$P = \begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

10. Show all your work.

Let α and β be the eigenvalues for A and B such that $\vec{v}$ is the corresponding eigenvector. We write $A\vec{v} = \alpha\vec{v}$ and $B\vec{v} = \beta\vec{v}$.

Then we have

$$\begin{aligned} (AB - BA)\vec{v} &= AB\vec{v} - BA\vec{v} \\ &= A(\beta\vec{v}) - B(\alpha\vec{v}) \\ &= \beta A\vec{v} - \alpha B\vec{v} \\ &= \beta\alpha\vec{v} - \alpha\beta\vec{v} = \vec{0} \end{aligned}$$

By the definition of the eigenvector $\vec{v} \neq \vec{0}$. Thus the matrix $AB - BA$ is not invertible.

$$\text{Therefore } \det(AB - BA) = 0$$

Extra writing space (Include the question number for each answer!)

Exercise 1

version
B

The parametric vector form for the general solution is:
 $x_3 \begin{pmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 1 \end{pmatrix}; x_3, x_5 \in \mathbb{R}$ OR $\text{Span} \left\{ \begin{pmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ OR other equivalent forms.

version
C

The parametric vector form for the general solution is:
 $x_3 \begin{pmatrix} -4 \\ 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}; x_3, x_5 \in \mathbb{R}$ OR $\text{Span} \left\{ \begin{pmatrix} -4 \\ 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

version
D

The parametric vector form for the general solution is:
 $x_3 \begin{pmatrix} -5 \\ 5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{pmatrix}; x_3, x_5 \in \mathbb{R}$ OR $\text{Span} \left\{ \begin{pmatrix} -5 \\ 5 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Correct answer

Exercise 4

version A

B

C

D

B

A

D

C

version A

version B

Exercise 6

(a)

$$\lambda_1 = -1 + 2i$$

$$\lambda_2 = -1 - 2i$$

$$\lambda_1 = -1 + 3i$$

$$\lambda_2 = -1 - 3i$$

version C

$$\lambda_1 = -1 + 4i$$

$$\lambda_2 = -1 - 4i$$

version C

$$\lambda_1 = -1 + 5i$$

$$\lambda_2 = -1 - 5i$$

(b) Same for all versions (see page 2)