

GENERAL ODE

Order: 1st or 2nd

Linearity: Linear or non-linear.

if linear: Homogeneous or non-homogeneous.

Coefficients: Constant or Variables.

Well-posedness or Ill-posedness

↳ BC and IC to unique solution.

LINEARITY → Proof that.

$$L(\alpha u + \beta v) = \alpha L u + \beta L v.$$

Non-LINEAR → Proof that

$$L(\alpha u) \neq \alpha L u$$

COMPLEX NUMBERS

$$\exp(x + iy) = \exp(x) \cdot [\cos(y) + i \sin(y)]$$

$$z = x + iy$$

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2}$$

$$\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2}$$

c.c. when $\bar{z} = x - iy$

Polar notation $p = \sqrt{x^2 + y^2}$
 $\phi = y/x$

$$z = p \exp(i\phi)$$

FORCED OSCILLATOR

$$m \frac{d^2 y}{dt^2} + r \frac{dy}{dt} + ky = k y_{\text{for}} \cos(\omega t)$$

$$\left\{ \frac{d^2 y}{dt^2} + 2\beta \omega_0 \frac{dy}{dt} + \omega_0^2 y = \omega_0^2 y_{\text{for}} \cos(\omega t) \right.$$

when $\beta = 0$; undamped

if $\omega \neq \omega_0 \rightarrow f_{\text{part}}(t) = C \cos \omega t$

if $\omega = \omega_0 \rightarrow f(t) = \frac{1}{2} y_{\text{for}} \omega_0 t \cdot \sin(\omega_0 t)$

when $\beta > 0$; damped

For damped case at equilibrium

$$y_{\text{eq}}(t) = \frac{1}{2} Y A_{\text{eq}} \exp(i\omega t) + \text{c.c.} \rightarrow \text{trial solution.}$$

↳ Replace in main equation to get A_{eq} .

$$\left\{ y_{\text{eq}}(t) = Y A_{\text{eq}} \cos(\omega t - \phi_{\text{eq}}) \rightarrow \phi_{\text{eq}} = \tan^{-1} \left\{ \frac{\text{Im}\{A_{\text{eq}}\}}{\text{Re}\{A_{\text{eq}}\}} \right\} \right.$$

RELAXATION AND GROWTH.

General form: $\frac{df}{dt} + af = b(t).$

Look for particular and homogeneous solution:

Then: $f(t) = f_{\text{hom}}(t) + f_{\text{part}}(t)$, match IC

For growth: $\frac{df}{dt} - af = b(t).$

SCALING AND TAYLOR'S EXPANSION.

For scaling consider. $x^* = \frac{x}{X}$, $t^* = \frac{t}{T}$

Use chain rule.

$$\frac{dx}{dt} = \frac{d(x^* X)}{dt^*} \cdot \frac{dt^*}{dt} = \frac{X}{T} \cdot \frac{dx^*}{dt^*}$$

$X \rightarrow \text{constant}$.

For Taylor's consider.

$$f(t+h) = f(t) + h \left. \frac{df}{dt} \right|_t + \frac{h^2}{2} \left. \frac{d^2 f}{dt^2} \right|_t + O(h^3)$$

• Consider $h = \Delta t$ or $h = -\Delta t$
Forward Backward.

for Forward $\frac{df}{dt} = \frac{f_{k+1} - f_k}{\Delta t}$

Stability is essential to analyze.

OSCILLATION

Helmholtz Problem:

$$\Delta^2 + \omega^2 = 0$$

General solution: $y(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t)$.

where $\Delta = \frac{dy}{dt}$.

Damping oscillator.

$$\frac{d^2 y}{dt^2} + 2\beta \omega_0 \frac{dy}{dt} + \omega_0^2 y = 0 \rightarrow \Delta^2 + 2\beta \omega_0 \Delta + \omega_0^2 = p(\Delta)$$

$$\text{Roots by } \Delta_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \omega_0 [-\beta \pm \sqrt{\beta^2 - 1}]$$

No Damping $\rightarrow \beta = 0 \rightarrow$ Helmholtz problem

Under damping $\rightarrow 0 < \beta < 1$ $\Delta_1 = \omega_0 - \beta$; $\Delta_2 = \omega_0 \sqrt{\beta^2 - 1}$

General solution $\rightarrow y(t) = \exp(\Delta_1 t) [A \cos(\Delta_2 t) + B \sin(\Delta_2 t)]$

Critical Damping $\rightarrow \beta = 1 \rightarrow \Delta = -\omega_0$

General solution $\rightarrow y(t) = (A + Bt) \exp(-\omega_0 t)$

Overdamping $\rightarrow \beta > 1$ $\Delta_{1,1}$ and $\Delta_{1,2}$

General solution $\rightarrow y(t) = A \exp(\Delta_{1,1} t) + B \exp(\Delta_{1,2} t)$

NUMERICAL APPROXIMATION. (CENTRAL DIFFERENCE)

$$\begin{aligned} f(t+h) &= f(t) + h \left. \frac{df}{dt} \right|_t + \frac{h^2}{2} \left. \frac{d^2f}{dt^2} \right|_t + O(h^3) & \text{Positive} \\ f(t-h) &= f(t) - h \left. \frac{df}{dt} \right|_t + \frac{h^2}{2} \left. \frac{d^2f}{dt^2} \right|_t + O(h^3) & \text{Negative} \end{aligned}$$

For 1st derivative sum expansions $\rightarrow$ divide by $2\Delta t$

For 2nd derivative substrat expansions - divide by $[\Delta t]^2$

GENERAL PDES

Classification $\rightarrow \frac{A \partial^2 u}{\partial x^2} + \frac{B \partial^2 u}{\partial x \partial t} + \frac{C \partial^2 u}{\partial t^2}$

Use $D_{iscr} = B^2 - 4AC$

if $\text{Disc} = 0 \rightarrow \text{Parabolic}$

:if $Disc_1 > 1$ - Hyperbolic

if $D_{\text{iscr}} < 1$ - Elliptic.

for scaling $\rightarrow$ remembers to define coefficient based on importance
e.g. Advection $>$ Diffusion

Advection - Diffusion equation

Imagine  A diagram of a cylinder. An arrow labeled 'F_{left}' points to the left face of the cylinder. An arrow labeled 'F_{right}' points to the right face of the cylinder. Below the left face is the label 'x_{left}' and below the right face is the label 'x_{right}'.

Total amount of $M - M_0 = \int_I c(x, t) dx$.

The change in time $\frac{\partial M}{\partial t} = F_{left} - F_{right} + P$
 $P \rightarrow$ Production.

Assuming F as differentiable, and $P = \int_I p \, dx$

$$\frac{\partial M}{\partial t} = \int_{\mathbb{R}} \left[\frac{\partial c}{\partial t} + \frac{\partial F}{\partial x} - p \right] dx = 0$$

For infinite space it should hold, then.

$$\frac{\partial c}{\partial t} + \frac{\partial F}{\partial x} = p \quad \text{considering } F = Y_c - \delta \frac{\partial c}{\partial x}$$

$$-\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} - D \frac{\partial^2 c}{\partial x^2} = p$$

advection diffusion.

BOUNDARY CONDITIONS

Dirichlet $\rightarrow u(0, t) = \psi_0(t), u(L, t) = \psi_L(t)$

Neumann $\rightarrow \partial u / \partial x(0, t) = \varphi_0(t), \partial u / \partial x(L, t) = \varphi_L(t)$

Robin $\rightarrow \alpha_0 u(0, t) + \beta_0 \partial_x u(0, t) = X_0(t) ; \alpha_L u(L, t) + \beta_L \partial_x u(L, t) = X_L(t)$

Spatially periodic $\rightarrow u(0, t) = u(L, t), \quad \partial u / \partial x(0, t) = \partial u / \partial x(L, t).$

STABILITY

Relaxation

$$R_{FE} = 1 - A ; R_{BE} = 1 / (1 + A)$$

Stable $\rightarrow |R| < 1$ Marginal $|R| = 1$

Unstable $\rightarrow |R| > 1$

Isolation

Stable $0 < w < 2$ Marginal $w = 2$

Unstable $w > 2$

Advection

C → Constant number

Stable	$C < 1$	Marginal	$C = 1$
--------	---------	----------	---------

Unstable $C > 1$

Diffusion

$D \rightarrow$ Diffusion number.

Stable $D < 1/2$

ADVECTION PROBLEM

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0 \quad \text{Apply moving coordinates } \xi = x - vt \\ \tau = t$$

then: $u(x,t) = \tilde{u}(\xi, \tau)$ for new equation

$$\frac{\partial u}{\partial t} = \frac{\partial \tilde{u}}{\partial \xi} \frac{d\xi}{dt} + \frac{\partial \tilde{u}}{\partial \tau} \frac{d\tau}{dt} = -v \frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \tilde{u}}{\partial \tau}$$

For both will lead to $\frac{\partial \tilde{u}}{\partial \tau} = 0$

General solution $\tilde{u}(\xi, \tau) = G(\xi) \rightarrow u(x,t) = G(x - vt)$

Final solution with IC and BC: if $x > vt \rightarrow u_0(x - vt)$
 $x < vt \rightarrow \psi(1 - x/v)$

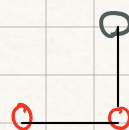
With relaxation $\frac{du}{dt} + v \frac{du}{dx} + au = 0$

Transforming coordinates $\frac{d\tilde{u}}{d\tau} + a\tilde{u} = 0 \rightarrow \text{solution } \tilde{u}(\xi, \tau) = G(\xi) \cdot \exp(-a\tau)$

UPWIND SCHEME

From Taylor's $\rightarrow \frac{du}{dx} \bigg|_{(x_j, t_k)} = \frac{u_{j,k} - u_{j-1,k}}{\Delta x}$ Note: use $j-1$, instead of j .

With forward Euler $\frac{du}{dt} \bigg|_{(x_j, t_k)} = \frac{u_{j,k+1} - u_{j,k}}{\Delta t}$ Stencil



For Non-linear advection consider: $\frac{du}{dt} + f(u) \frac{du}{dx} = 0$

DIFFUSION PHENOMENA

$$\frac{\partial u}{\partial t} = \delta \frac{\partial^2 u}{\partial x^2} \quad \text{Parabolic}$$

In equilibrium $\rightarrow \frac{d^2 u_{eq}}{dx^2} = 0 \rightarrow u_{eq} = Cx + D$ (C and D from BC)

ANALYTICAL SOLUTION

For homogeneous problem

Separation of variables $\cdot u(x,t) = X(x) \cdot T(t)$
 into PDE gives $X(x) \cdot T'(t) = \delta X''(x) \cdot T(t) \rightarrow \frac{T'(t)}{\delta T(t)} = \frac{X''(x)}{X(x)} = -K^2$

From that consider uncoupled PDEs

$$X''(x) = -X(x)K^2 \rightarrow \text{Helmholtz} \rightarrow X(x) = A \cos(Kx) + B \sin(Kx)$$

$$T'(t) = -T(t)\delta K^2 \rightarrow \text{relaxation} \rightarrow T(t) = \exp(-K^2 \delta t)$$

Next $\rightarrow$ Apply BC and later IC for final solution.

NUMERICAL APPROXIMATION.

Use FTCS,

Forward time

Central space

~ Forward Euler

~ Central Space Difference

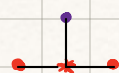
• FE.

$$\frac{du}{dt} = \frac{u_{j,k+1} - u_{j,k}}{\Delta t}$$

• CS

$$\frac{d^2 u}{dx^2} = \frac{u_{j+1,k} - 2u_{j,k} + u_{j-1,k}}{(\Delta x)^2}$$

Stencil



* Evaluated.
• Calculated

WAVE PHENOMENA

NOT IN THE EXAM

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{Hyperbolic.}$$

Using moving coordinates
$$\begin{cases} \xi = x - vt \\ \eta = x + vt \end{cases}$$

Leads to $u(x, t) = \tilde{u}(\xi(x, t), \eta(x, t))$

With chain rule.

$$\frac{\partial u}{\partial x} = \frac{\partial \tilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \tilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \tilde{u}}{\partial \eta}$$

2nd order derivative, so:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial x} \right) \cdot \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial x} \right) \cdot \frac{\partial \eta}{\partial x}$$

} Essential steps

$$\begin{aligned} & \frac{\partial}{\partial \xi} \left(\frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \tilde{u}}{\partial \eta} \right) \cdot 1 + \frac{\partial}{\partial \eta} \left(\frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \tilde{u}}{\partial \eta} \right) \cdot 1 = \frac{\partial^2 \tilde{u}}{\partial \xi^2} + \frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \tilde{u}}{\partial \eta^2} \end{aligned}$$

Doing it for time and space gives $\frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} = 0$

which is solved by $\tilde{u}(\xi, \eta) = G(\xi) + H(\eta) \rightarrow u(x, t) = G(x - vt) + H(x + vt)$.

For arbitrary functions G and H .

By separation of variables $u(x, t) = X(x) \cdot T(t)$

Similar to diffusion $\frac{T''}{T} = \frac{X''}{X} = -k^2$ which gives Helmholtz problem.

$$\rightarrow u(x, t) = (A \cos(kx) + B \sin(kx)) \cdot (C \cos(\gamma k t) + D \sin(\gamma k t))$$

Solved by applying BC and IC.

Damped harbour problem consider

$$\frac{\partial \xi}{\partial t} + M \frac{\partial u}{\partial x} = 0; \quad \frac{\partial u}{\partial t} + g \frac{\partial \xi}{\partial x} + \frac{r u}{M} = 0$$

Can be transformed as
$$\frac{\partial^2 \xi}{\partial t^2} - g M \frac{\partial^2 \xi}{\partial x^2} + \frac{r}{M} \frac{\partial \xi}{\partial t} = 0$$

Undamped when $r=0$

leads to
$$\xi(x, t) = \sum_{\cos k_0 L}^{\cos k_0 x} \cdot \cos \omega t.$$

For Numerical solution use $\rightarrow$ CTCS (Central time central space).