

Course : Calculus 2	Date : Monday 3 April, 2023
Module : 2A	Time : 8:45 – 10:45 (2 hours)
Course code : 202001218	Reference : ME-CAL2-2223-exam

Calculus 2

Exam

Instructions

This exam contains 10 exercises. You shall use the provided *answer form* to submit your answers.

- For exercises 1–6, you are only required to fill in the **final answer** on the answer form.
- For exercises 7–10, you are required to write down a **full calculation and argumentation**.

You will hand in your answer form only. **Any text outside the answer form will not be considered.**

If you run out of space, you can use the extra space at the end of the answer form. Refer clearly to that space in the original answer.

*** The use of electronic devices is not allowed. ***

Final answer questions

1. Below there are three statements. Which of these three statements is/are **True**? [2 pt]
Report your answer on the answer form.

- (1) A particle moves in space in such a way that its position at time t is given by

$$r(t) = a \cos(t) \mathbf{i} + a \sin(t) \mathbf{j} + bt \mathbf{k}, \quad \text{where } a \text{ and } b \text{ are real numbers.}$$

The speed of the particle at $t = 2$ is $\sqrt{a^2 + 4b^2}$.

- (2) Let f be a differentiable function from $\mathbb{R}^2$ into $\mathbb{R}$. The derivative at $P = (x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j}$ is

$$D_{\mathbf{u}}f(x_0, y_0) = |\nabla f(x_0, y_0)| \sin(\theta),$$

where θ is the angle between the vectors $\nabla f(x_0, y_0)$ and $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j}$.

- (3) Let $f(x, y, z)$ be a differentiable function. The tangent plane at the point $P_0 = (x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ is

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0.$$

- | | |
|---|--|
| <p>☐ A (1), (2), and (3) all are True</p> <p>☐ B none of them is True</p> <p>☐ C (2) and (3) only are True</p> <p>☐ D (1) and (3) only are True</p> | <p>☐ E (1) and (2) only are True</p> <p>☐ F (1) only is True</p> <p>☐ G (2) only is True</p> <p>☐ H (3) only is True</p> |
|---|--|

2. Fill in your final answer on the answer form.

- (a) Given the equation $y + e^x + \ln(y + 1) = x + y^3$, and assuming y as a differentiable function of x , find $\frac{dy}{dx}$. [2 pt]

- (b) Let $f(x, y, z) = (xy + yz)^2$. Let $x = \cos(u - 1) + \sin(v - 1)$, $y = u^2 + v^2$ and $z = u + v$. [2 pt]
Use the chain rule to evaluate the partial derivative of f with respect to u at $u = 1$ and $v = 1$.

3. Fill in your final answer on the answer form.

Let $f(x, y)$ be a function. The following information is provided:

$$\frac{\partial f}{\partial x} = \frac{1}{x^2} \quad \frac{\partial f}{\partial y} = -\frac{2}{y^2} \quad \text{and} \quad f(1, 1) = 1.$$

(a) Let $L(x, y)$ denote the linearization of $f(x, y)$ at $(1, 1)$. Determine $L(x, y)$. [1 pt]

(b) Let $P(x, y)$ denote the second degree Taylor polynomial of $f(x, y)$ at $(1, 1)$. Determine $P(x, y)$. [1 pt]

4. The double integrals [3 pt]

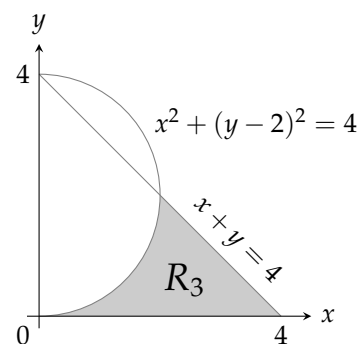
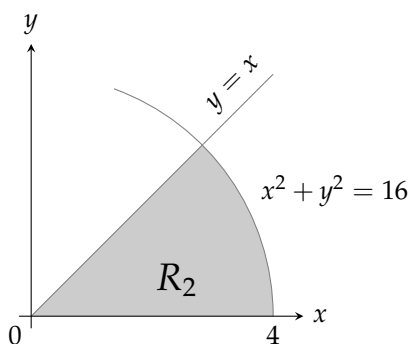
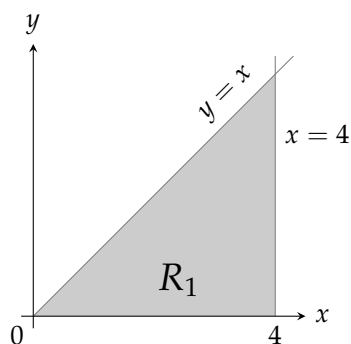
$$\iint_{R_1} y \, dA, \quad \iint_{R_2} y \, dA \quad \text{and} \quad \iint_{R_3} y \, dA,$$

are taken over the shaded regions R_1 , R_2 and R_3 shown in the figure.

Each double integral has **only one** matching expression among the options A—F provided below (this could be *either* in Cartesian *or* in polar coordinates).

Determine the matching expression for each integral.

Report your answer on the answer form.



[A] $\int_0^2 \int_{\sqrt{4y-y^2}}^{4-y} y \, dx \, dy$

[B] $\int_0^{2\sqrt{2}} \int_y^{4-y^2} y \, dx \, dy$

[C] $\int_0^4 \int_0^y y \, dx \, dy$

[D] $\int_0^{\pi/4} \int_0^4 r \sin \theta \, dr \, d\theta$

[E] $\int_0^{2\sqrt{2}} \int_y^{\sqrt{16-y^2}} y \, dx \, dy$

[F] $\int_0^{\pi/4} \int_0^{4/\cos \theta} r^2 \sin \theta \, dr \, d\theta$

5. Consider the region R of the Cartesian plane bounded by the graph of $y = \ln x$, the x -axis and the line $x = e$. Let $f(x, y) = 4x$.

(a) Express $\iint_R f(x, y) \, dA$ as an iterated integral, using the order of integration $dx \, dy$. [2 pt]

(b) Compute $\iint_R f(x, y) \, dA$. [1 pt]

(c) Find the average value of f over R . [1 pt]

Fill in your final answer on the answer form.

6. Let D be the region in the first octant, bounded by the planes $y = 0$, $x = y$ and $z = 0$, and by the sphere of radius 3 centered at the origin. Express the volume of D as an iterated triple integral in spherical coordinates, using the order of integration $d\rho d\phi d\theta$. [1 pt]
 Fill in your final answer on the answer form. Do not evaluate the integral.

Open questions

7. The full solution to this exercise must be clearly written down in the answer form (including calculations and argumentation). [5 pt]

Let $f(x, y) = 2xy^2 - 12x^2 - 6y^2 + 2$.

- Determine the critical point (or points) of $f(x, y)$.
- Find the local maximum, local minimum values and saddle points of the function (if any).

8. The full solution to this exercise must be clearly written down in the answer form (including calculations and argumentation). [5 pt]

Use the method of Lagrange multipliers to find the extreme values of the function $f(x, y, z) = 4x + 2y + 2z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 5$.

9. The full solution to this exercise must be clearly written down in the answer form (including calculations and argumentation).

A thin plate on the Cartesian plane is bounded by the four lines:

$$y = -x + 1, \quad y = -x + 3, \quad y = \frac{1}{4}x - 4, \quad y = \frac{1}{4}x + 4.$$

- (a) Find the Jacobian $J(u, v)$ of the transformation given by $u = x + y$ and $v = y - \frac{1}{4}x$. [2 pt]

- (b) Find the total mass of the plate, if its density is given by the function $\delta(x, y) = \frac{1}{2}x + \frac{1}{2}y$. [3 pt]

10. The full solution to this exercise must be clearly written down in the answer form (including calculations and argumentation).

A solid of constant density $\delta = 1$ is bounded above by the surface $z = 3 \cdot \sqrt{1 - \sqrt{x^2 + y^2}}$

and below by the surface $z = \sqrt{1 - \sqrt{x^2 + y^2}}$.

- (a) Write the triple integral in cylindrical coordinates describing the volume (and hence the mass) of the solid. *You don't need to evaluate the integral.* [2 pt]

- (b) Find the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the solid, knowing that its total mass is $M = \frac{16}{15}\pi$. [3 pt]
 Write a full calculation and argumentation in the answer sheet.

Total: 36 pt

Answer Form

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Full name : _____

Student number : _____

▲ ▲ ▲ DO NOT FORGET TO FILL IN YOUR DATA ▲ ▲ ▲

Question 1. Mark your answer with a cross.

☐ A ☐ B ☐ C ☐ D ☐ E ☐ F ☐ G ☐ H

Question 2(a). $\frac{dy}{dx} =$

Question 2(b). $\frac{\partial f}{\partial u}$ at $u = 1$ and $v = 1$ is

Question 3(a). $L(x, y) =$

Question 3(b). $P(x, y) =$

Question 4. Mark your answer with a cross.

$$\iint_{R_1} y \, dA = \begin{array}{|c|} \hline \text{A} \\ \hline \end{array} \begin{array}{|c|} \hline \text{B} \\ \hline \end{array} \begin{array}{|c|} \hline \text{C} \\ \hline \end{array} \begin{array}{|c|} \hline \text{D} \\ \hline \end{array} \begin{array}{|c|} \hline \text{E} \\ \hline \end{array} \begin{array}{|c|} \hline \text{F} \\ \hline \end{array}$$

$$\iint_{R_2} y \, dA = \begin{array}{|c|} \hline \text{A} \\ \hline \end{array} \begin{array}{|c|} \hline \text{B} \\ \hline \end{array} \begin{array}{|c|} \hline \text{C} \\ \hline \end{array} \begin{array}{|c|} \hline \text{D} \\ \hline \end{array} \begin{array}{|c|} \hline \text{E} \\ \hline \end{array} \begin{array}{|c|} \hline \text{F} \\ \hline \end{array}$$

$$\iint_{R_3} y \, dA = \begin{array}{|c|} \hline \text{A} \\ \hline \end{array} \begin{array}{|c|} \hline \text{B} \\ \hline \end{array} \begin{array}{|c|} \hline \text{C} \\ \hline \end{array} \begin{array}{|c|} \hline \text{D} \\ \hline \end{array} \begin{array}{|c|} \hline \text{E} \\ \hline \end{array} \begin{array}{|c|} \hline \text{F} \\ \hline \end{array}$$

Question 5(a). $\iint_R f(x, y) \, dA$ as an iterated integral, using the order of integration $dx \, dy$, is

Question 5(b). $\iint_R f(x, y) \, dA =$

Question 5(c). The average value of f over R is

Question 6. The iterated triple integral in spherical coordinates, using the order of integration $d\rho \, d\varphi \, d\theta$, is

[illegible]

Question 8. Write a full calculation/argumentation clearly in the box.

[illegible]

Question 9(a). Write a full calculation/argumentation clearly in the box.

[illegible]

Question 9(b). Write a full calculation/argumentation clearly in the box.

[illegible]

[illegible]

This image shows a single sheet of white paper with horizontal blue or grey ruling lines. The lines are evenly spaced and run across the width of the page. There is no handwriting or other markings on the paper.

[illegible]

[illegible]