

Linear Algebra_A April 03, 2023

Answer sheet Answer the questions legibly in the spaces provided. If you run out of room for an answer, continue on the back of the answer sheet by clearly referring to the question.

Name and Student Number: _____

Programme: _____

Question	1	2	3	4	5	6	7	8	9	10	Total
Points Possible	2	2	3	2+1+3	3+2	2	3	3	4	3+3	36
Points Earned											

1. Which of the options is the augmented matrix of the given system?

☐ (A)
☐ (B)

correct : B

☐ (C)
☐ (D)

2. Add a check mark (✓) for each option in the reduced echelon form.

versions

correct A, B, D, F, H

☐ (C)
☐ (D)

☐ (E)
☐ (F)

☐ (G)
☐ (H)

3. Show all your work.

let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$

The equation holds if

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_1 + y_2 \end{pmatrix}$$

Then $\begin{cases} x_1 = -y_2 \\ x_2 = y_1 + y_2 \end{cases}$ or $\begin{cases} y_1 = x_1 + x_2 \\ y_2 = -x_1 \end{cases}$

4. a) Only provide your final answer.

$$\vec{x} = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, x_4 \in \mathbb{R} \quad \text{or} \quad \vec{x} = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

OR $\vec{x} \in \left\{ \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}, t \in \mathbb{R}$

- b) Only provide your final answer.

$$\text{Null } A = \text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

or

$$\text{Null } A = \left\{ \vec{x} \mid \vec{x} = k \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, k \in \mathbb{R} \right\}$$

- c) Show all your work.

From the given augmented matrix R , we can observe that the first three columns have pivots.

A basis for $\text{col } A$ can be written as the set of the first three column vectors of A .

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \right\}$$

Moreover, to obtain a basis that includes $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, we must ensure that the set has three vectors and the two vectors we choose are linearly independent from each other and from $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

Such a basis can be written as $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Alternatively: $\text{col } A = \mathbb{R}^3$ therefore a basis that includes $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ can be $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

5. a) Only provide your final answer.

correct k: | All k ∈ ℝ except for
0 or 2

- b) Only provide your final answer.

i) $\det B^T A$

-6

ii) $\det B^{-1}$

1/3

iii) $\det A^3$

-8

iv) $\det 5A$

$5^3 * (-2)$
 $= -250$

6. Only provide your final answer.

correct : 15

7. Show all your work.

We know that $A^{-1} A = A A^{-1} = I$

$$A A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad-bc} \begin{pmatrix} ad-bc & -ab+ab \\ cd-cd & -cb+da \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{ad-bc} & 0 \\ 0 & \frac{ad-bc}{ad-bc} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I. \text{ Therefore } A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

8. Show all your work.

To find the eigenvalues of $5I + 3A$, let's consider $\vec{x}$ with the right dimension,

$$(5I + 3A)\vec{x} = 5I\vec{x} + 3A\vec{x} = 5\vec{x} + 3A\vec{x}$$

Since the eigenvalues of A are $-1, 0, 1$ then
 $\{A\vec{x} = -1\cdot\vec{x} \text{ or } A\vec{x} = 0\cdot\vec{x} \text{ or } A\vec{x} = 1\cdot\vec{x}\}$

Therefore $5\vec{x} + 3A\vec{x} = 5\vec{x} - 3\vec{x} = 2\vec{x}$ or

$$5\vec{x} + 3A\vec{x} = 5\vec{x} - 0\vec{x} = 5\vec{x} \text{ or}$$

$$5\vec{x} + 3A\vec{x} = 5\vec{x} + 3\vec{x} = 8\vec{x}$$

Therefore, the eigenvalues of $5I + 3A$ are 2, 5, and 8.

9. Show all your work.

$$\det \begin{pmatrix} 3-\lambda & 0 & 0 \\ -3 & 4-\lambda & 9 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (3-\lambda)(4-\lambda)(3-\lambda), \lambda_{1,2}=3, \lambda_3=4$$

$$N - 3I = \begin{pmatrix} 0 & 0 & 0 \\ -3 & 1 & 9 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_3 = \text{Null}(N - 3I) = \text{Span} \left\{ \begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ or equivalent}$$

$$N - 4I = \begin{pmatrix} -1 & 0 & 0 \\ -3 & 0 & 9 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_4 = \text{Null}(N - 4I) = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

therefore

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1/3 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

10. a) Show all your work.

$$\begin{aligned} \text{If } T\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \text{ then by linearity } T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \\ T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) &= \frac{1}{2}T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) - T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \frac{1}{2}\left(\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ +1/2 \\ 1 \end{pmatrix} \\ T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) &= T\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}\right) + T\left(\begin{pmatrix} 0 \\ x_2 \end{pmatrix}\right) = x_1 T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + x_2 T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= x_1 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ +1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ 2x_1 + x_2/2 \\ x_2 \end{pmatrix} \end{aligned}$$

b) Show all your work.

The representation matrix of T is

$$A = \begin{pmatrix} 0 & 2 \\ 2 & 1/2 \\ 0 & 1 \end{pmatrix}$$

i) $A \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\text{Null } A = \{\vec{0}\}$ therefore T is one-to-one.

ii) $\text{Im } T \neq \mathbb{R}^3$ or $\text{col } A \neq \mathbb{R}^3$ therefore T is not onto.

$$(\text{col } A = \text{span} \left\{ \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1/2 \\ 1 \end{pmatrix} \right\} \neq \mathbb{R}^3)$$