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A

Answer Form

Calculus 1A – Exam – Friday 27 October, 2023

Full name in BLOCK LETTERS _____

Student number _____

Programme _____

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Question 1(a). Write only your answer in the frame below.

$$\frac{1}{3} \langle 1, 1, 4 \rangle \text{ or } \frac{1}{3} \hat{i} + \frac{1}{3} \hat{j} + \frac{4}{3} \hat{k} \text{ or } \begin{pmatrix} 1/3 \\ 1/3 \\ 4/3 \end{pmatrix}$$

Question 1(b). Write only your answer in the frame below.

$$\begin{aligned} 3x + y + z &= 2 \\ \text{or} \\ z &= 2 - 3x - y \\ \text{or} \\ 3x + y + z - 2 &= 0 \end{aligned} \quad \text{or} \quad \begin{cases} a = 3 \\ b = 1 \\ c = 1 \\ d = 2 \end{cases}$$

Question 1(c). Write only your answer in the frame below.

$$\theta = \frac{\pi}{3} \text{ or } 60^\circ$$

Question 1(d). Write only your answer in the frame below.

$$\langle \frac{1}{2}, 2, \frac{1}{2} \rangle \text{ or } \frac{1}{2} \hat{i} + 2 \hat{j} + \frac{1}{2} \hat{k} \text{ or } \begin{pmatrix} 1/2 \\ 2 \\ 1/2 \end{pmatrix}$$

Question 2. Write a full motivation/argumentation in the frame below.

$$f(x) = \begin{cases} \frac{\sqrt{x}-2}{x-4} & \text{for } x > 4 \\ a & \text{for } x \leq 4 \end{cases}, \text{ where } a \in \mathbb{R} \text{ and } x \in \mathbb{R}.$$

The function is defined at $x=4$. We need to consider both one-sided limits.

$$\lim_{x \rightarrow 4^-} a = a$$

#1 (see alternative solution below)

$$\begin{aligned} \lim_{x \rightarrow 4^+} \frac{\sqrt{x}-2}{x-4} &= \lim_{x \rightarrow 4^+} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x \rightarrow 4^+} \frac{\cancel{(x-4)}}{\cancel{(x-4)}(\sqrt{x}+2)} \\ &= \lim_{x \rightarrow 4^+} \frac{1}{\sqrt{x}+2} = \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \text{"0/0"-type} &= \lim_{x \rightarrow 4^+} \frac{\frac{1}{2\sqrt{x}}}{1} = \frac{1}{4} \\ \text{apply L'Hopital} & \end{aligned}$$

Both limit techniques #1 and #2 are accepted.

Finally, we conclude that f is continuous at $x=4$ if

$$\lim_{x \rightarrow 4^-} a = \lim_{x \rightarrow 4^+} \frac{\sqrt{x}-2}{x-4} = f(4) = \frac{1}{4}$$

$$\text{So } a = 1/4$$

Question 3(a). Write only your answer in the frame below.

$$x > 0 \quad \text{or} \quad x \in (0, \infty) \quad \text{or} \quad (0, \infty)$$

Question 3(b). Write a full motivation/argumentation in the frame below.

$$h(x) = \frac{1}{x} + \ln(x) \longrightarrow h'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{-1+x}{x^2}$$

So $h'(x) = 0$ if and only if $x = 1$

$h'(x)$ is undefined if and only if $x^2 = 0$, so $x = 0$.
However, $x = 0$ is not part of the function's domain.
Therefore, the only critical point happens at $x = 1$.

Question 3(c). Write a full motivation/argumentation in the frame below.

The critical point of h on $[\frac{1}{2}, 4]$ is only $x = 1$.
Other candidates for extreme values are the boundaries of $[\frac{1}{2}, 4]$.

Note: $h(1/2) = -0.7$ and $h(4) = 1.38$

Unfortunately, the hint does not have accurate interval. However, it does not affect the actual answer.

x	$h(x)$
$1/2$	$2 + h(1/2) \cong 1.3$
1	1
4	$1/4 + h(4) \cong 1.6$

The absolute maximum of $h(x)$ is $1/4 + h(4)$ and it's attained at $x = 4$. The absolute minimum of $h(x)$ is 1 and it's attained at $x = 1$.

Question 4. Write a full motivation/argumentation in the frame below.

Note : There is a typo, it should be $(e^x - 1)$ instead of $(1 - e^x)$.

$$\lim_{x \rightarrow 0^+} \frac{\ln(e^x - 1)}{2 \ln(x)}, \quad \frac{\infty}{\infty} \text{-type}$$

L'Hopital Rule

$$\lim_{x \rightarrow 0^+} \frac{\frac{e^x}{e^x - 1}}{2 \cdot \frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{x \cdot e^x}{2(e^x - 1)} \quad \frac{0}{0} \text{-type}$$

Apply again L'Hopital Rule

$$\lim_{x \rightarrow 0^+} \frac{e^x + x e^x}{2 e^x} = \frac{1}{2}$$

Question 5(a). Write a full motivation/argumentation in the frame below.

If approaching $(0,0)$ along a line is used: let $y = mx$ for some value of m , then

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along line} \\ y=mx}} \frac{3xy}{x^2+2y^2} = \lim_{x \rightarrow 0} \frac{3mx^2}{x^2+2m^2x^2} = \lim_{x \rightarrow 0} \frac{3m}{1+2m^2}$$

So for instance, if (x,y) is on the x -axis, then $y=0$, which can be achieved by choosing $m=0$. In that case, the limit is 0 .

If (x,y) is on the line $y=x$, then $m=1$, and consequently the limit is 1 .

These results differ and hence the limit of the function at $(0,0)$ does not exist.

Question 5(b). Write a full motivation/argumentation in the frame below.

Since the limit does not exist at $(0,0)$, the function is not continuous at $(0,0)$.

Question 6. Write a full motivation/argumentation in the frame below.

$$f(x,y) = \sqrt{y-x^2}$$

The equation for the tangent plane through $(a,b,f(a,b))$ is

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) \quad (*)$$

where $(a,b) = (1,2)$.

Calculate the partial derivatives of f :

$$\frac{\partial f}{\partial x}(x,y) = \frac{-x}{\sqrt{y-x^2}} \longrightarrow \frac{\partial f}{\partial x}(1,2) = -1$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{2\sqrt{y-x^2}} \longrightarrow \frac{\partial f}{\partial y}(1,2) = 1/2$$

Fill out all results in $(*)$

$$z = 1 - 1(x-1) + \frac{1}{2}(y-2)$$

This can be written as

$$z = -x + \frac{y}{2} + 1$$

or

$$x - \frac{y}{2} + z = 1$$

or

$$x - \frac{y}{2} + z - 1 = 0$$

Extra writing space. Clearly refer to this space in the original answer.

Version A

1. This is a final answer question. Provide your final answer (and only your final answer) on the answer sheet. Do not use this sheet.

Define $P(1, -1, 0)$, $Q(2, 0, 4)$, and $R(2, 3, 1)$. Let $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PR}$.

- (a) Determine vector $\mathbf{w}$ which is in the direction of $\mathbf{v}$ and has the magnitude $|\mathbf{w}| = \sqrt{2}$. [1 pt]
 (b) The point P lies in the plane W . It is also given that the plane W is parallel to the plane V given by $3x + y + z = 7$. Determine an equation for the plane W of the form $ax + by + cz = d$. [1 pt]
 (c) Calculate the angle between $\mathbf{u}$ and $\mathbf{v}$. [2 pt]
 (d) Calculate the vector projection of $\mathbf{u}$ onto $\mathbf{v}$. [2 pt]

(1a) $\bar{\mathbf{w}} = \frac{1}{3} \langle 1, 1, 4 \rangle$

(1b) $3x + y + z = 2$

(1c) $\theta = \pi/3$

(1d) $\text{proj}_{\mathbf{v}} \mathbf{u} = \langle 1/2, 2, 1/2 \rangle$

Version B

1. This is a final answer question. Provide your final answer (and only your final answer) on the answer sheet. Do not use this sheet.

Define $P(-1, 1, 2)$, $Q(1, 3, 1)$, and $R(2, 1, -1)$. Let $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PR}$.

- (a) Determine vector $\mathbf{w}$ which is in the direction of $\mathbf{u}$ and has the magnitude $|\mathbf{w}| = 2$. [1 pt]
 (b) The point P lies in the plane W . It is also given that the plane W is parallel to the plane V given by $x + 4y - z = 10$. Determine an equation for the plane W of the form $ax + by + cz = d$. [1 pt]
 (c) Calculate the angle between $\mathbf{u}$ and $\mathbf{v}$. [2 pt]
 (d) Calculate the vector projection of $\mathbf{u}$ onto $\mathbf{v}$. [2 pt]

(1a) $\bar{\mathbf{w}} = \frac{2}{3} \langle 2, 2, -1 \rangle$

(1b) $x + 4y - z = 1$

(1c) $\theta = \pi/4$

(1d) $\text{proj}_{\mathbf{v}} \mathbf{u} = \langle 3/2, 0, -3/2 \rangle$

Extra writing space. Clearly refer to this space in the original answer.

Version C

1. This is a final answer question. Provide your final answer (and only your final answer) on the answer sheet. Do not use this sheet.

Define $P(1, -2, 0)$, $Q(2, -1, 4)$, and $R(2, 2, 1)$. Let $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PR}$.

- (a) Determine vector $\mathbf{w}$ which is in the direction of $\mathbf{v}$ and has the magnitude $|\mathbf{w}| = \sqrt{2}$. [1 pt]
- (b) The point P lies in the plane W . It is also given that the plane W is parallel to the plane V given by $2x - 3y + z = 8$.
Determine an equation for the plane W of the form $ax + by + cz = d$. [1 pt]
- (c) Calculate the angle between $\mathbf{u}$ and $\mathbf{v}$. [2 pt]
- (d) Calculate the vector projection of $\mathbf{v}$ onto $\mathbf{u}$. [2 pt]

$$(1a) \quad \bar{\mathbf{w}} = \frac{1}{3} \langle 1, 4, 1 \rangle$$

$$(1b) \quad 2x - 3y + z = 8$$

$$(1c) \quad \theta = \pi/3$$

$$(1d) \quad \text{proj}_{\mathbf{u}} \mathbf{v} = \langle 1/2, 1/2, 2 \rangle$$

Version D

1. This is a final answer question. Provide your final answer (and only your final answer) on the answer sheet. Do not use this sheet.

Define $P(-1, 6, 2)$, $Q(1, 4, 3)$, and $R(0, 6, 3)$. Let $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PR}$.

- (a) Determine vector $\mathbf{w}$ which is in the direction of $\mathbf{u}$ and has the magnitude $|\mathbf{w}| = 2$. [1 pt]
- (b) The point P lies in the plane W . It is also given that the plane W is parallel to the plane V given by $-2x + y - z = 15$.
Determine an equation for the plane W of the form $ax + by + cz = d$. [1 pt]
- (c) Calculate the angle between $\mathbf{u}$ and $\mathbf{v}$. [2 pt]
- (d) Calculate the vector projection of $\mathbf{v}$ onto $\mathbf{u}$. [2 pt]

$$(1a) \quad \bar{\mathbf{w}} = \frac{2}{3} \langle 2, -2, 1 \rangle$$

$$(1b) \quad -2x + y - z = 6$$

$$(1c) \quad \theta = \pi/4$$

$$(1d) \quad \text{proj}_{\mathbf{u}} \mathbf{v} = \langle 2/3, -2/3, 1/3 \rangle$$

Q5a: Alternative solution

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} \frac{3xy}{x^2+2y^2} &= \lim_{r \rightarrow 0^+} \frac{3r^2 \sin \theta \cos \theta}{r^2 \cos^2 \theta + 2r^2 \sin^2 \theta} \\ &= \lim_{r \rightarrow 0^+} \frac{3 \sin \theta \cos \theta}{\cos^2 \theta + 2 \sin^2 \theta}\end{aligned}$$

This limit depends on θ ; for $\theta = 0$, the limit is 0

and for $\theta = \pi/4$, the limit is 1.

These results differ and hence the limit of the function at $(0,0)$ does not exist.