

1.) area = $\int_C f(x,y) ds = (*)$

parametrization: $r(t) = \cos(t)i + \frac{1}{2}\sin(t)j, 0 \leq t \leq 2\pi$

velocity: $r'(t) = -\sin(t)i + \frac{1}{2}\cos(t)j$

speed: $|r'(t)| = \sqrt{\sin^2(t) + \frac{1}{4}\cos^2(t)}$

f on C : $f(r(t)) = \sqrt{\frac{1}{4}\cos^2(t) + \sin^2(t)}$

$$\begin{aligned} \rightarrow (*) &= \int_0^{2\pi} \sin^2(t) + \frac{1}{4}\cos^2(t) dt \\ &= 2\pi - \frac{3}{4} \int_0^{2\pi} \cos^2(t) dt \\ &= \frac{5\pi}{4} \end{aligned}$$

$= \frac{1}{2} \left(t + \cos(t)\sin(t) \right) \Big|_0^{2\pi} = \pi$

2.a) Component test: F is conservative $\Leftrightarrow \text{curl } F = 0$.

$$\text{curl } F = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2(x-1)e^{y^2+z^2} & 2(x-1)^2 y e^{y^2+z^2} & 2(x-1)^2 z e^{y^2+z^2} + z \end{vmatrix}$$

$$= \left(4yz(x-1)^2 e^{y^2+z^2} - 4yz(x-1)^2 e^{y^2+z^2} \right) \mathbf{i} - \left(4(x-1)z e^{y^2+z^2} - 4z(x-1) e^{y^2+z^2} \right) \mathbf{j}$$

$$+ \left(4y(x-1) e^{y^2+z^2} - 4y(x-1) e^{y^2+z^2} \right) \mathbf{k} = 0,$$

i.e. F is conservative.

b) potential f : $F = \nabla f$

$$(i) \quad 2(x-1)e^{y^2+z^2} = \frac{\partial f}{\partial x} \Rightarrow f(x,y,z) = (x-1)^2 e^{y^2+z^2} + g(y,z)$$

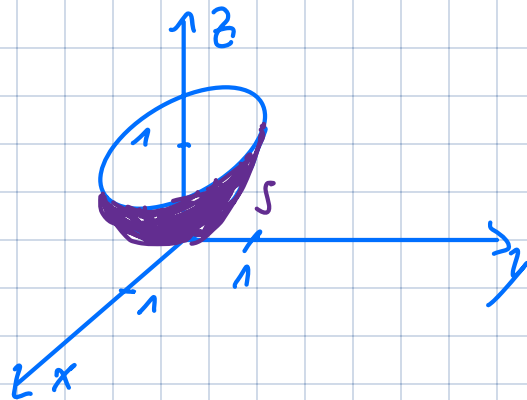
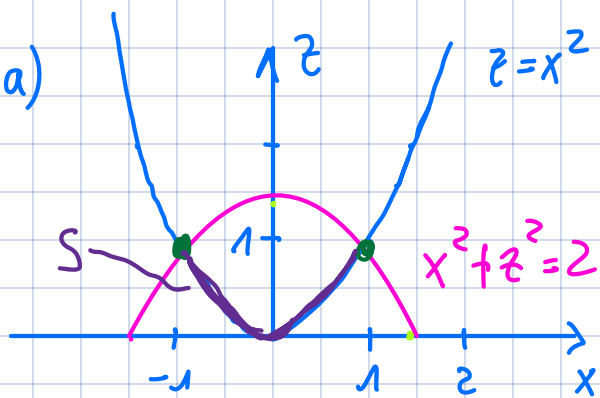
$$(ii) \quad 2(x-1)^2 y e^{y^2+z^2} = \frac{\partial f}{\partial y} = 2y(x-1)^2 e^{y^2+z^2} + \frac{\partial g}{\partial y} \Rightarrow g(y,z) = h(z)$$

$$(iii) \quad 2(x-1)^2 z e^{y^2+z^2} + z = \frac{\partial f}{\partial z} = 2z(x-1)^2 e^{y^2+z^2} + h'(z) \Rightarrow h'(z) = z$$

$$\Rightarrow f(x,y,z) = (x-1)^2 e^{y^2+z^2} + \frac{1}{2} z^2$$

$$\begin{aligned} c) \quad \text{work} &= \int_C F \cdot T \, ds \\ &= f(B) - f(A) \\ &= 1/2. \end{aligned}$$

3.a)



b)
$$\text{area} = \iint_S d\sigma = (*)$$

parametrization:
$$\vec{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r^2 \mathbf{k}$$

with
$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 1$$

[note: on intersection: $z = 2 - z^2$, which is satisfied for $z=1, z=-2 \rightarrow r=\sqrt{z}=1$]

surface area element
$$d\sigma = |\vec{r}_r \times \vec{r}_\theta| dr d\theta$$

$$\vec{r}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + 2r \mathbf{k}$$

$$\vec{r}_\theta = -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j}$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = -2r^2 \cos \theta \mathbf{i} - 2r^2 \sin \theta \mathbf{j} + r \mathbf{k}$$

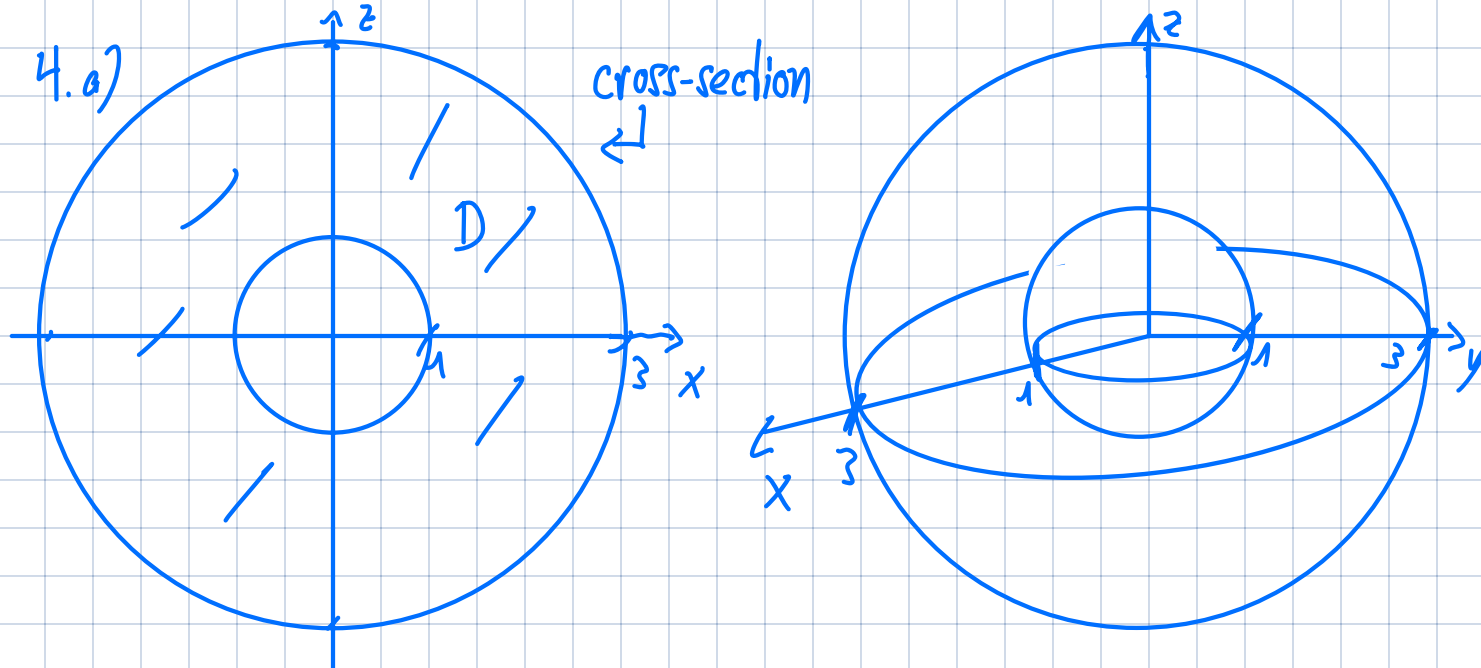
$$|\vec{r}_r \times \vec{r}_\theta| = r \sqrt{1 + 4r^2}$$

$$\Rightarrow (*) = \int_0^{2\pi} \int_0^1 r \sqrt{1 + 4r^2} dr d\theta$$

$$= 2\pi \int_1^5 \frac{1}{8} \sqrt{u} du$$

$$u = 1 + 4r^2$$

$$= \frac{\pi}{6} \left(5^{\frac{3}{2}} - 1 \right)$$



parametrization: $x = \rho \cos \theta \sin \varphi$

$$y = \rho \sin \theta \sin \varphi$$

$$z = \rho \cos \varphi$$

with $0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi, 1 \leq \rho \leq 3$

b) flux = $\iint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, d\sigma$

$$= \iiint_D \underbrace{\operatorname{div}(\mathbf{F})}_{=6} \, dV$$

$$= 3x^2 + \underbrace{\cos x}_{=1} \underbrace{\cos z}_{=1} + 6y^2 + 3x^2 + 6z^2 - \underbrace{\cos x}_{=1} \underbrace{\cos z}_{=1} = 6\rho^2$$

$$= 6 \int_0^{2\pi} \int_0^\pi \int_1^3 \rho^2 \rho^2 \sin \varphi \, d\varphi \, d\theta \, d\rho$$

$$= 12\pi \underbrace{\left(-\cos \varphi \right)_0^\pi}_{=2} \left(\frac{1}{5} \rho^5 \right)_1^3 = \frac{24\pi}{5} (3^5 - 1)$$